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GREEK MATHEMATICS

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This chapter discusses the tradition of theoretical mathematics that formed a part of Greek literary culture. This was not the only kind of mathematics that existed in the ancient Greek world. There were also traditions of elementary school mathematics, and the subscientific traditions of mathematics that were handed down by various professionals who used mathematics in their work, such as tradesmen, builders, accountants, and astrologers.[1] In fact, in the ancient Mediterranean, these subliterary traditions almost certainly formed the vast majority of all the mathematics studied and practiced, while literary, theoretical mathematics was practiced by only a privileged few.[2] Nevertheless, the elite, literary status of the theoretical mathematicians, along with the brilliance of their work, ensured that much of their project was preserved for posterity, while the predominantly oral, subscientific tradition survives only in scattered, material fragments and a few collections passed down through the manuscript tradition, attributed, probably erroneously, to Heron (ca. mid-first to third century CE).

In the classical period, when the forms of literary mathematical texts were being established, elite, theoretical mathematicians appear to have done as much as they could to separate themselves from professionals who used mathematics, just as they strove to distinguish themselves from other elites, such as sophists and philosophers who did not engage in mathematical activity. Theoretical mathematics was originally not a professional, institutionalized activity. During the Hellenistic period, when this, now more

[1] The term "subscientific" is due to J. Høyrup. See, for example, J. Høyrup, "Sub-scientific Mathematics: Observations on a Pre-Modern Phenomena," *History of Science* 27 (1990), 63–87. The social context of such "subscientific" mathematics is treated by S. Cuomo, *Ancient Mathematics* (Routledge: London, 2001).

[2] M. Asper, "The Two Cultures of Mathematics in Ancient Greece," in E. Robson and J. Stedall (eds.), *The Oxford Handbook of the History of Mathematics* (Oxford: Oxford University Press, 2008), pp. 107–32.

institutionalized, theoretical mathematics was applied to serious problems in natural science and engineering, there were a number of attempts to wed the theoretical and practical traditions of mathematics, but the social context and institutional settings of the two remained distinct. Finally, in the Imperial and Late Ancient periods, although creative mathematics was less practiced, mathematical scholarship was thoroughly institutionalized in the philosophical schools, and mathematics and philosophy were finally united, although for many mathematical scholars of the late period, philosophy was accorded a superior position.

The range of ideas and activities that were designated by the word *mathēmatikē* are not identical to those denoted by our word *mathematics*.[3] From the earliest times, *mathēmatikē* was connected with any branch of learning, but came to denote the mathematical sciences centered around arithmetic, geometry, astronomy, and harmonics. From the time of the Pythagoreans (ca. late sixth to fifth century BCE) to that of Ptolemy (ca. mid-second century CE), astronomy and harmonics were not regarded as applications of mathematics, but as core areas of the enterprise. *Mathēmatikē* eventually denoted those literary disciplines that used mathematical techniques or investigated mathematical objects, whether actual or ideal, and which included fields such as optics, sphere-making, or astrology along with abstract investigations such as the theories of whole numbers or conic sections.

Nevertheless, ancient thinkers had a fairly clear idea of what constituted a mathematical work and who was a mathematician. They described different arrangements of the mathematical sciences and set out various relationships between mathematical fields and other branches of theoretical knowledge. Ancient discussions of mathematicians revolve around a core group of frequently repeated names.[4] We find debates about the legitimacy of certain mathematical arguments in philosophical works, but rarely in the surviving mathematical texts. Mathematicians defined mathematics by the nature of the texts they wrote. From around the middle of the fifth century BCE, mathematical texts were highly structured, involving explicit arguments, often centered around diagrams and letter-names or other technical apparatus, and mathematicians were those people who could produce such texts. They developed particular ways of speaking, or rather of writing, and these further served to reinforce the exclusive tendencies of this small literary group.

[3] See G. E. R. Lloyd, "Methods and Problems in the History of Ancient Science," *Isis* 83 (1992), 564–77, 569–70.

[4] Netz identified 144 mathematicians in ancient sources, many of whom are only known to us by name. See R. Netz, "Classical Mathematicians in the Classical Mediterranean," *Mediterranean Historical Review* 12 (1997), 1–24.

OBSCURE ORIGINS

The origins of Greek mathematics, either in Greek philosophy or as an independent discipline, used to be a favorite topic for historians of Greek mathematics.[5] The difficulty is that we have little certain evidence about the details of these origins. Most of our evidence for this early work comes from writings that were produced centuries later, or through the filter of philosophical writings that were not intended to be of historical value and were irregular in their use of mathematical prose. One of our most important sources for this early period is the so-called catalog of geometers, which appears as a passage in Proclus' *Commentary on Elements Book I*.[6] Parts of this passage are thought to go back to a now lost *History of Geometry* written by Aristotle's student and colleague, Eudemus of Rhodes (ca. late fourth century BCE). Proclus (412–85 CE), however, does not explicitly attribute this passage to Eudemus, in contrast to other borrowings, and it has clearly been modified by other authors over the years. Moreover, this passage does not provide much description of the mathematical activities of the individuals it names and, hence, serves us mostly as a relative chronology. The earlier evidence we have for the origins of Greek mathematics, such as in the writings of Plato and Aristotle, is often vague, not attributing work to individual mathematicians, or presenting incomplete arguments.[7] It should be clear from this that our most expansive sources are far removed in time from the persons and events they recount and must be treated with due caution.

The images of Thales and Pythagoras writing detailed mathematical proofs now seem to be the stuff of legends. When we consider how late our sources for this activity are, the other writings that remain from this early period by their contemporaries, and the tendency of Greek doxographers to produce rationalized histories by associating well-known results with well-known thinkers, it appears increasingly unlikely that the early Presocratics or Pythagoreans produced writings containing deductive mathematics. Although Thales and the early Pythagoreans may well have made various correct assertions about elementary geometry and some sort of argument for why these were true, the proofs attributed to them by Proclus are now

[5] For examples, see A. Szabó, *The Beginnings of Greek Mathematics*, trans. A. M. Ungar (Dordrecht: Reidel, 1978), and W. Knorr, *The Evolution of the Euclidean Elements* (Dordrecht: Reidel, 1975).

[6] G. Friedlein, *Procli Diadochi in Primum Euclidis Elementorum Librum Commenatarii* (Leipzig: Teubner, 1873). For a recent discussion of the catalog of geometers, see L. Zhmud, *The Origin of the History of Science in Classical Antiquity* (Berlin: Walter de Gruyter, 2006), pp. 179–90.

[7] For the mathematics contained in Plato and Aristotle, see R. S. Brumbaugh, *Plato's Mathematical Imagination* (Bloomington, IN: Indiana University Press, 1954) and T. Heath, *Mathematics in Aristotle* (Bristol: Thoemmes Press, 1949). For examples of the difficulties involved in interpreting the mathematics in Plato and Aristotle, see F. Acerbi, "Plato: Parmenides 149a7–c3. A Proof by Complete Induction?," *Archive for History of Exact Sciences* 55 (2000), 57–76, and H. Mendell, "Two Traces of Two-Step Eudoxan Proportion Theory in Aristotle: A Tale of Definitions in Aristotle, with a Moral," *Archive for History of Exact Sciences* 61 (2007), 3–37.

generally thought to have been reconstructed by later ancient authors.[8] Although Pythagoras almost certainly had a profound interest in numbers, there is no early evidence that he or the early Pythagoreans produced deductive mathematics.[9] The earliest Pythagoreans of whom we can develop a clear picture are Philolaus of Croton (ca. turn of the fourth century BCE) and Archytas of Tarentum (ca. mid-fifth to mid-fourth centurries BCE) – but the mathematical work of Philolaus is rather meager, while Archytas is already a contemporary of Plato's and his mathematics does not appear to have been uniquely Pythagorean.[10]

Although much of the early history of Greek mathematics was probably reconstructed by Eudemus and other authors on the basis of vague attributions and the guiding belief that mathematics must develop in a rational way, in the case of Hippocrates of Chios (ca. mid-fifth century BCE), Eudemus seems to have had some written sources. Indeed, Simplicus (ca. early sixth century CE), in his *Commentary on Aristotle's Physics*, tells us that Eudemus regarded Hippocrates as one of the earliest mathematicians.[11] We are told that Hippocrates worked on the problem of doubling the cube and reduced this to the more general problem of constructing two mean proportionals between two given lines,[12] and was the first to write up the principles of geometry in an *Elements*.[13] Most importantly, however, we have a long fragment of his work preserved by way of Eudemus' *History of Geometry*. In this passage, he shows how to square three different types of lunes, in an attempt to use these figures to reduce the problem of squaring a circle to something more manageable.[14] From these writings we learn that at this time there were certain problems that were considered worthy of solution and that Hippocrates approached these using a general strategy of reducing them to problems that were soluble by constructions made on lettered diagrams. From the details of his arguments, we see that the types of

[8] Proclus, in his *Commentary to Elements Book I*, assigns *Elements* I 15, 26 to Thales and *Elements* I 32 and 42 to the Pythagoreans. See Friedlein, *Procli Diadochi*.

[9] W. Burkert argued that there is no early evidence for mathematical activity by Pythagoras himself and that there was nothing uniquely Pythagorean about the mathematics practiced by later members of the school. See W. Burkert, *Lore and Science in Ancient Pythagoreanism* (Cambridge, MA: Harvard University Press, 1972). A case that the early Pythagoreans worked in deductive mathematics was remade by L. Zhmud, *Wissenschaft, Philosophie und Religion im frühen Pythagoreismus* (Berlin: Bücher, 1997), but has not gained much acceptance.

[10] For Philolaus and Archytas, see C. Huffman, *Philolaus of Croton* (Cambridge: Cambridge University Press, 1993), and C. Huffman, *Archytas of Tarentum* (Cambridge: Cambridge University Press, 2005).

[11] H. Diels, *Simplicii in Aristotelis Physicorum libros quattor priores commentaria* (Berlin: Remer, 1882), p. 69.

[12] For example, Eutocius, *Commentary on Archimedes' Sphere and Cylinder*. As for Eratosthenes, see J. L. Heiberg, *Archimedis opera omnia cum commentariis Eutocii* (Leipzig: Teubner, 1910–15), vol. 3, 88.

[13] This comes from Proclus in *Commentary to Elements Book I*. See Friedlein, *Procli Diadochi*, p. 66.

[14] W. Knorr, *The Ancient Tradition of Geometric Problems* (Boston, MA: Birkhäuser, 1989), pp. 29–39, and R. Netz, "Eudemus of Rhodes, Hippocrates of Chios and the Earliest Form of a Greek Mathematical Text,: *Centaurus* 46 (2004), 243–86.

constructions allowed and the starting points of the argument were still taken at the geometer's discretion.

Another important development in the fifth century was the discovery of incommensurability – that is, the realization that there are cases where two given magnitudes have no common measure.[15] For example, the combination of Pythagorean interests in number theory and various geometric studies came together in deductive arguments that the diagonal of a square is incommensurable with its side.[16] This, in turn, led to considerable interest in incommensurability and excited the activity of some of the best mathematicians of the fifth and fourth centuries: Theodorus (ca. late fifth century BCE); Eudoxus (ca. mid-fourth century BCE); and Theaetetus (ca. early fourth century to 369 BCE). What we think we know about the work of Theodorus is the product of repeated reconstructions by modern scholars based on a few tantalizing hints found mostly in Plato's writings.[17] In the case of Eudoxus and Theaetetus, however, some of their work is considered to have formed the substantial basis of *Elements* V and X, respectively, and hence to form the foundations of ratio theory and to provide the most complete surviving study of incommensurability.

In the fourth century, there were a number of important mathematicians who were traditionally associated with Plato's Academy. Although it is now clear that the image of Plato as an organizer of mathematical activity is more a product of the scholars of the early Academy than a reflection of reality,[18] Plato's writings are full of mathematical references and claims about the benefits of a mathematical education. Nevertheless, there is no reason to believe that Plato had more influence on his mathematical colleagues than they had on him. On the contrary, it seems clear that Plato used the ideas that he learned from Theodorus, Archytas, and Theaetetus to develop his own philosophy.

Whatever the case, this period of Athenian cultural influence acted as catalyst for significant mathematical work. Solutions to the problem of duplicating the cube, in Hippocrates' reduction to the problem of finding two mean proportionals, were put forward by Archytas, Eudoxus, and Menaechmus (ca. mid-fourth century BCE).[19] Archytas produced theorems in number theory that may have formed the basis of *Elements* VIII and the Euclidean *Section of the Canon*.[20] A theory of conic sections was developed

[15] Knorr dates this discovery to 430–410 BCE. See Knorr, *Evolution*, p. 40.
[16] Knorr, *Evolution*, pp. 21–8.
[17] Ibid., pp. 62–130.
[18] L. Zhmud, "Plato as 'Architect of Science'," *Phronesis* 43 (1998), 211–44.
[19] Knorr, *Ancient Tradition*, pp. 50–76.
[20] Knorr, *Evolution*, pp. 211–25. Note, however, that the attribution to Archytas of *Elements* VIII and *Section of the Canon* is rather tenuous. See C. Huffman, *Archytas*, pp. 451–70.

by Eudoxus, Menaechmus, and others and applied to the solution of various problems. It is difficult to appraise how general or complete this theory was.

Eudoxus was one of the most brilliant mathematicians of this period. He developed a ratio theory that was closely related to the ratio theory we now read in *Elements* V, but which may have relied on a two-stage argument, showing first the case of commensurable magnitudes and then that of incommensurable magnitudes.[21] Although this work has generally been read as an attempt to make a general theory of ratio, it is not a complete foundation for contemporary or later mathematical practice and hence can also be interpreted as a collection of theorems useful for geometry.[22] Eudoxus also wrote a number of theorems concerning the mensuration of objects that were praised by Archimedes and which probably involved double indirect arguments. In astronomy, he put forward the two-sphere model of the cosmos with a spherical earth in the center of a spherical firmament, and a model of homocentric spheres to account for some set of celestial phenomena; unfortunately the details of these works are a matter of speculation.[23]

By the time Alexander began his conquests, Greek mathematics had undergone considerable development. Of the three great problems of antiquity – the quadrature of a circle, the trisection of an angle, and the duplication of a cube – the first two had been clearly articulated and addressed and the duplication of the cube had been solved. Although the details of their content are matters of reconstruction and speculation, general theories of number, ratio, and incommensurable magnitudes had been developed, and texts treating the elements of geometry were in circulation. We do not know the precise relationship between these early works and Euclid's *Elements*, but there is no direct indication in the sources that early geometers were interested in restricting the set of constructions that could be carried out to those that are abstractions from a straightedge and compass.

GEOMETRY IN THE HELLENISTIC PERIOD

Around the beginning of the Hellenistic period there were a number of projects to consolidate and reformulate the considerable body of

[21] W. Knorr, "Archimedes and the Pre-Euclidean Proportion Theory," *Archive for History of Exact Sciences* 28 (1978), 183–244, F. Acerbi, "Drowning by Multiples," *Archive for History of Exact Sciences* 57 (2003), 218–24, and Mendell, "Two Traces of Two-Step Eudoxan Proportion Theory."

[22] K. Saito, "Phantom Theories of Pre-Eudoxean Proportion," *Science in Context* 16 (2003), 331–47.

[23] There is disagreement about how Eudoxus' model functioned, and even about what phenomena it was supposed to model. See, for examples, B. Goldstein and A. Bowen, "A New View of Greek Astronomy," *Isis* 74 (1983), 330–40, H. Mendell, "The Trouble with Eudoxus," in P. Suppes, J. V. Moravcsik, and H. Mendell (eds.), *Ancient and Medieval Traditions in the Exact Sciences* (Stanford, CA: CLSL Publications, 2000), pp. 59–138, and I. Yavetz, "A New Role for the Hippopede of Eudoxus," *Archive for History of Science* 56 (2001), 69–93.

mathematical knowledge that Greek scholars had produced. At the early Lyceum, Eudemus of Rhodes wrote his histories of geometry, arithmetic, and astronomy.[24] Although it is not clear how many written sources he had for the earliest periods, his work became a major source for the later estimation of these fields. Around the turn of the century, or shortly after, Euclid (ca. early third century BCE), the most widely read mathematician of the ancient period, undertook a project of giving a solid foundation to, and a clear articulation of, nearly all branches of the exact sciences.

We know nothing of Euclid's life. It is often assumed that he worked at Alexandria, due to the circulation of a legend that he told King Ptolemy that there is no royal road to geometrical knowledge,[25] and to Pappus' statement that Apollonius worked with his students in Alexandria.[26] Claims that he worked at the Museum, or the Library, are simply embellishments of these two hints. What we do know is that he produced a considerable body of mathematical work, the influence of which increased throughout the Hellenistic period and which had become canonical by the Imperial period.

Euclid worked widely in nearly every area of mathematics and the exact sciences. Most of these works were concerned with geometry or the application of geometry to natural science. Although Euclid will always be known as the author of the *Elements*, he also wrote works on conic sections, solid loci, and porisms, and other specialized works related to geometrical analysis. He was regarded by Pappus (ca. early fourth century CE) as one of the three primary authors of the field of geometrical analysis, along with Aristaeus (ca. mid-fourth to mid-third century BCE) and Apollonius (ca. late third century BCE).[27] In the exact sciences, he wrote works on spherical astronomy, mechanics, optics, catoptrics, and harmonic theory, although the authenticity of the last three has been questioned.[28] Because of the fame of the *Elements*, and because his more advanced mathematical works have been lost, modern scholars have often seen Euclid as a mere compiler and textbook writer, not a productive mathematician. This was not, however, the assessment of ancient mathematicians, such as Apollonius and Pappus. Nevertheless, it is clear that Euclid had a great interest in codifying mathematical knowledge and setting it on a secure foundation.

We see this initially in his division of geometry into different domains based on the types of techniques that can be used and, more explicitly, in his

24 For a full study of Eudemus' work, see Zhmud, *The Origin of the History of Science*.

25 There are, in fact, two versions of this legend, both from late authors: one involves Euclid and King Ptolemy, while the other involves Menaechmus and Alexander the Great. See Proclus, *Commentary on Elements Book I*, chap. 4, and Stobaeus, *Anthology* II 31.115.

26 Pappus, *Collection* VII 35. See F. Hultsch, *Pappi Alexandrini Collectionis* (Berlin: Weidmann, 1876) and A. Jones, *Pappus of Alexandria, Book 7 of the Collection* (New York: Springer, 1986).

27 Pappus, *Collection* VII 1. See Jones, *Book 7 of the Collection*, p. 83.

28 These questions of authenticity are highly subjective, often relying on dubious arguments such as claims that an author who produced the *Elements*, which has so many good arguments in it, could not also have produced the *Section of the Canon*, which contains some sloppy arguments.

arrangement of the *Elements*. The *Elements* begins with a series of definitions, postulates, and common notions, some of which may not have been authentic,[29] but most of which almost certainly were. In particular, the postulates, which deal with construction, right angles, and parallelism, are logically required by the development of the book. The line and circle construction techniques and the parallel postulate, along with the logical structure developed on the basis of these, may well have been Euclid's original contribution to the foundations of geometry. It is clear from what Aristotle says that the theory of parallel lines was still plagued with logical issues when he was writing,[30] and we find no mention of the construction postulates in authors before Euclid and no interest in a restriction to these constructions in authors who lived around his time, such as Autolycus, Aristarchus (both ca. early third century BCE), or Archimedes (280s–212 BCE). Hence, the *Elements* can be read as a treatise which develops mathematics along the Aristotelian principle of showing as much as possible on a limited set of starting points.

The early books of the *Elements*, I–VI, treat plane geometry; however, this includes *Elements* V on theory of ratios between magnitudes, where magnitude is understood as a hyperonym, or the abstraction of a feature, of geometric objects – the length of a line, the area of a figure, the volume of a solid, and so forth. The next three books, VII–IX, treat number theory, and *Elements* X deals with incommensurability, which arises as a topic when we try to apply the theory of numbers to magnitudes, such as lines and areas. The final three books develop a theory of solid geometry.

It has often been remarked that *Elements* V and X do not fit very well into this general plan, and hence must have been taken essentially unaltered from previous works. It is also possible, however, to read these books as being in their proper place in the overall deductive structure.[31] If we read the text as having a single architecture, Euclid's strategy appears to have been to introduce an idea or theory as late as possible, so as to show how much can be done without it. Once it became necessary to introduce new concepts and methods, however, Euclid does not treat them as purely instrumental, but presents them in the broader context of an articulated theory. This strategy can be exemplified by the first book. The structure of *Elements* I shows that congruency of triangles can be shown independently of the parallel postulate while equality of areas cannot.[32]

[29] For a discussion of the authenticity of the early definitions, see L. Russo, "The Definitions of Fundamental Geometric Entities Contained in Book I of Euclid's Elements," *Archive for History of Exact Sciences* 52 (1998), 195–219.

[30] Heath, *Mathematics in Aristotle*, pp. 27–30.

[31] For a discussion of the deductive structure of the *Elements* see I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, MA: MIT Press, 1981). Mueller, however, does not read the text as having a single deductive structure and alters the ordering of the subject matter.

[32] The theory of congruency does, however, contain another unstated axiom in the construction of *Elements* I 16. This is the assumption that a line can always be extended so as to be longer than any given length, which does not hold on a surface of positive curvature, such as a sphere.

The so-called Pythagorean theorem, *Elements* I 47, is a culmination of the book, but we should not understand it as *the* goal. The book has various goals, such as the production of constructive methods for geometric problem-solving, the articulation of a theory of congruency that is only loosely related to constructive geometry, the foundation of a theory of parallelism, and its development into a theory of area, which results in *Elements* I 47 and in the theory of the application of areas found in *Elements* II.

Another major area of Euclid's mathematical endeavor was in what came to be known as the "field of analysis." Although most of this work has been lost,[33] it formed an important foundation for advanced research in mathematics and contributed to Euclid's high reputation as a mathematician in antiquity. Along with the *Data*,[34] he produced a *Conics*, which was later superseded by that of Apollonius; a short work on the *Division of Figures*;[35] a treatise on loci that form surfaces; and a treatise called *Porisms*, treating problem-like propositions that make assertions about what objects or relations will be given if certain conditions are stated about given objects.[36] While the *Data* was probably written more as a repository of useful results than as a treatise to be read from beginning to end, the *Solid Loci* and *Porisms* may have been written to teach students useful habits of thought in the analytical approach to problems. Whatever the case, by the Imperial period, these works were taken by Pappus, and other teachers of advanced mathematics, as part of a canon for the study of analysis.

In contrast to Euclid, Archimedes is the ancient mathematician that we know the most about.[37] From his own writings we learn that his father was an astronomer,[38] that he was associated with the court of Hieron II of Syracuse, and that he kept in regular contact with his mathematical colleagues working in Alexandria. Because of his iconic status in antiquity, many legends circulated about Archimedes, most of which are probably apocryphal.[39] Many of these tales concern his death, and many of them are likely to have been exaggerated as well. Nevertheless, it seems clear that he was an old man when he was killed by Roman solders during the sack of Syracuse under Marcus Claudius Marcellus, in 212 BCE.[40] It is also probable

33 Only the *Data* survives in Greek, along with a fragment of the *Division of Figures*, in Arabic.

34 See H. Menge (ed.), *Euclidis Data cum commentario Marini et scholiis antiquis* (Leipzig: Teubner, 1896) (*Euclidis Opera omnia*, vol. 6).

35 J. Hogendijk, "The Arabic version of Euclid's *On Divisions*," in M. Folkerts and J. P. Hogendijk (eds.), *Vestigia Mathematica* (Amsterdam: Rodopi, 1993), pp. 143–62.

36 For a discussion of the lost treatises on analysis by Euclid and others, see Jones, *Book 7 of the Collection*, pp. 510–619.

37 The standard work on his life and work is E. J. Dijksterhuis, *Archimedes*, trans C. Dikshoorn (Princeton, NJ: Princeton University Press, 1987).

38 This is based on a plausible conjecture for a meaningless passage in our received manuscripts. See F. Blass, "Der Vater des Archimedes," *Astronomische Nachrichten* 104 (1883), 255.

39 Archimedes was the archetypal Greek mathematician for ancient authors. See M. Jaeger, *Archimedes and the Roman Imagination* (Ann Arbor, MI: University of Michigan Press, 2008).

40 Plutarch, *Life of Marcellus*, 14–19, and Polybius, *Histories*, 8.5–8.8.

that the siege of the city was prolonged due to engines of war that Archimedes built, although none of these are likely to have been burning mirrors. Moreover, from the prefaces that he wrote to his works, we are able to get a sense of his personality. He comes off as having been justly proud of his abilities and aware of the fact that there were very few who could really appreciate what he was doing. He also, however, had a playful side and sometimes teased the Alexandrian mathematicians by sending them false propositions and encouraging them to find the proofs.[41] This playfulness also extended to his own work as we see from his analysis of an ancient game, the *Stomachion*, and his work the *Sand Reckoner*, in which he shows that his system of large numbers, which cannot be represented in the normal Greek number system, can handle computing the number of grains of sand that it would take to fill even Aristarchus' absurdly large universe, in which the earth is assumed to orbit around the sun.[42]

The relationship between Archimedes and Euclid is difficult to untangle. Archimedes may have been Euclid's younger contemporary, but since Euclid's dates are only vaguely known, it is also possible that Euclid died some years before Archimedes was born. Any direct influence from Euclid's work on Archimedes is also difficult to detect. On the one hand, Archimedes appears to have been unimpressed by Euclid's foundational approach, showed no interest in a restriction to line and circle constructions, and occasionally proved things using different techniques from those we find in the *Elements*. On the other hand, he does not repeat material we find in Euclid's works and it may be that, like a number of other great mathematicians, he came to appreciate Euclid's project more as he got older and was increasingly influenced by the *Elements* throughout the course of his long career.[43] Proclus' claim that Archimedes mentions Euclid used to be explained away by pointing out that a reference to *Elements* I 2 in *Sphere and Cylinder* I 2 is an obvious interpolation; however, in the so-called Archimedes Palimpsest there is another reference to Euclid elsewhere in *Sphere and Cylinder* that may be authentic.[44]

Archimedes' works can be divided into three types: geometrical; mechanical; and computational. In all three of these, however, we can see certain features that are characteristic of Archimedes' personal style. They tend to be

[41] Heiberg, *Archimedes*, vol. 2, 2–4.

[42] B. E. Wall, "The Historiography of Aristarchos of Samos," *Studies in the History and Philosophy of Science* 6 (1975), 201–88, and R. Netz, "The Goal of Archimedes' *Sand-Reckoner*," *Apeiron* 36 (2003), 251–90.

[43] W. Knorr used the different ways in which Archimedes employs Euclid's ratio theory to provide a dating of his works. See W. Knorr, "Archimedes and the Elements: Proposal for a Revised Chronological Ordering of the Archimedean Corpus," *Archive for History of Exact Sciences* 19 (1978), 211–90. See, also, B. Vitrac, "A propos de la chronologie des oeuvres d'Archimède," in J.-Y. Guillaumin (ed.), *Mathématiques dans l'Antiquité* (Saint-Étienne: L'Université Saint-Étienne, 1992), pp. 59–93.

[44] Friedlein, *Procli Diadochi*, pp. 68; R. Netz, W. Noel, N. Wilson, and N. Tchernetska (eds.), *The Archimedes Palimpsest*, 2 vols. (Cambridge: Cambridge University Press, 2011), vol. 2, 276–7.

short, never more than two books. They cover distinct problems, or areas, and start with axioms and construction techniques suitable to the task at hand, with little interest in reducing these to more elementary concepts or methods. They show an abiding interest in mensuration, often in the form of a numerical comparison between the properties of a well-known object and those of a less tractable one. On the whole, Archimedes wrote advanced mathematical texts for mathematicians and, with the notable exception of his *Method*, seemed to have had little interest in questions of pedagogy or foundations.

Archimedes wrote no elementary geometric treatises. He wrote a text *On Spiral Lines*, which defined the curves by moving points, set out some of their properties, based on arithmetic progressions and *neusis* constructions involving setting a given length between given objects, and concluded by finding a number of significant areas related to the curves. In *On Conoids and Spheroids*, he opens with a series of definitions and a discussion of the problems to be addressed, proves a number of theorems on arithmetic series and the areas of ellipses, and then enters the main body of the work, in which he treats the volumes of various conics of revolution and their sections. One of his most striking works is *On the Sphere and Cylinder*. The first book is a systematic treatment of the relationships between various objects – like a circle and polygons that inscribe and circumscribe it, a cone and pyramids that inscribe and circumscribe it, a sphere and the series of sections of a cone that inscribe and circumscribe it, and so on – which leads to finding key relationships between a sphere and a cone, or cylinders, which inscribe or circumscribe it. The second book then uses this material to solve a number of difficult problems and to show theorems dealing with the volume and area of segments of a sphere.

Archimedes' surviving works in mechanics have to do with statics and the equilibrium of floating bodies. In *On the Equilibria of Planar Figures*, he demonstrates the principle of the balance, which appears to have played a major role in his research. In the so-called *Method*, he explains to his correspondent Eratosthenes (ca. mid-third to early second century BCE) how he used the idea of a virtual balance to derive many of his important mensuration results, found in works like *Quadrature of the Parabola* and *Conoids and Spheroids*, using the notion of suspending what we could call infinitesimals so as to maintain equilibrium on a balance.[45] Using as an example a hoof-shaped object that he had not treated previously,[46] in *Method* 12–15, Archimedes shows how one can first investigate the solid heuristically using the virtual balance, then develop a proof strategy using indivisibles, and finally write a fully rigorous proof using a double indirect

[45] E. Hayashi and K. Saito, *Tenbin no Majutsushi: Arukimedesu no Sūgaku* (Tokyo: Kyoritsu Shuppan, 2011).

[46] A section of a right cylinder formed by a plane passing through the diameter of the base.

argument. Although the *Method* was not known before 1906 and appears to have exerted no influence on the development of mathematics, it is an invaluable glimpse into the working habits of antiquity's greatest mathematician.

Probably the most widely read of Archimedes' computational works was the *Measurement of the Circle*. Although our current version of the text appears to be an abridged and highly edited epitome of Archimedes' orginal work, it provides us with at least part of his general approach to the classical problem of squaring the circle.[47] It begins by showing that the area of a circle is equal to a triangle whose base is the circumference of the circle and whose height is the radius of the circle, and it concludes with a general, although slow, iterative method for approximating the value of π by inscribing and circumscribing a circle with a polygon. In *Measurement of the Circle*, this process ends with the 96-gon; however, Heron, in his *Measurements*, states that Archimedes produced an even more precise set of bounds, but there are sufficient difficulties with the numbers in the received manuscripts to have generated a good deal of discussion but little agreement.[48]

Archimedes, unlike Euclid, neither wrote for beginners nor made grand efforts towards systematization. Instead, he wrote monographs addressing specific sets of problems or areas of theory that were susceptible to mensuration but were difficult enough to demonstrate his considerable abilities. When the necessity for a rigorous presentation required that he include trivial material, we can sense his boredom and haste, but when he ventures into uncharted waters he is careful to present a thorough proof, occasionally giving two arguments for the same result.[49]

The most successful mathematician of the next generation was Apollonius; however, he appears to have followed more closely in the tradition of Euclid than that of Archimedes. The only indication that we have of Apollonius continuing an Archimedean project is Pappus' description in *Collection* II of Apollonius' system of large numbers. On the whole, however, Pappus' statement that Apollonius studied with the students of Euclid in Alexandria seems to fit well with his mathematical style.[50] His works are both systematic and comprehensive. He had an interest in laying the foundation of the areas in which he worked and in providing elementary texts for more advanced mathematics, such as the theory of conic sections

[47] For the history of the text, see W. Knorr, *Textual Studies in Ancient and Medieval Geometry* (Boston, MA: Birkhäuser, 1989), pp. 805–16.

[48] There are as many reconstructions for these values and their derivation as there are scholars who have studied the matter. W. Knorr provides a summary of previous work, followed by his own reconstruction. See W. Knorr, "Archimedes and the Measurement of the Circle: A New Interpretation," *Archive for History of Exact Sciences* 15 (1975), 115–40.

[49] Examples of the former are *Sphere and Cylinder* I 28–32 and *Spheroids and Conoids* 11, and of the latter *Sphere and Cylinder* II 8.

[50] It is also possible, however, that the idea that Apollonius studied with Euclid's students was simply an inference by Pappus, or someone else, based on the similarity of their mathematical style.

and the field of analysis. Like Euclid, he divided his texts according to the constructive approaches that were used: line and circles; conic sections; and general curves.[51]

Apollonius' most significant work was his *Conics*, the first four books of which provided "a training in the elements" of conic sections,[52] and the latter four exploring more advanced topics that were of use in geometric analysis.[53] The final book of the treatise, in which Apollonius showed how his theory could be used to solve interesting problems, has been lost since ancient times. The entire book was clearly written to be of use in geometrical analysis: in the introductions to the individual books, Apollonius explains how the material he covers will serve the student of analysis, and Pappus includes the *Conics* as the last work in his treatment of the field of analysis. The elementary books were apparently based on Euclid's work in conic theory, although Apollonius redefined the conic sections, derived the principle properties (*symptōmata*) by a construction from any initial cone, gave a more general treatment, and furnished a number of theorems that were not known to Euclid. The more advanced books are said to be an addition to the work of various predecessors, but again Apollonius points out how he clarified, simplified, and extended their contributions.

The treatise begins by showing the analogies that exist between various types of conic sections such as triangles, circles, parabolas, ellipses, and hyperbolas,[54] and how the fundamental properties of the three final conic sections can be derived from relationships between certain straight lines related to the original cone. After developing the basic theory of tangents, *Conics* I ends by providing constructions of the conic sections given certain straight lines that involve finding the cone, of which the sought conic is a section. *Conics* II then treats diameters and asymptotes, and ends with a series of problems involving tangents. Book III provides a sort of metrical theory of conic sections and their tangents that shows the invariance of certain areas constructed with parallels to the tangents, and which Apollonius regarded as essential for a complete solution to the three and four line locus problems. The final books deal with topics like the intersections of various conic sections, minimum and maximum lines, equal and similar sections, and the diameters of conic sections and the figures that contain them. In each case, we see that the material is related to certain types

[51] See Pappus, *Collection* IV 36, for a discussion of the division into three classes of problems, or domains of geometry.

[52] J. L. Heiberg, *Apollonii Pergaei quae graece exstant cum commentariis antiquis*, 2 vols. (Leipzig: Teubner, 1891–3), vol. 1, 2–4.

[53] For a discussion of the contents of the *Conics*, see H. G. Zeuthen, *Die Lehre von den Kegelshnitten in Altertum* (Copenhagen: Andr. Fred. Höst & Sohn, 1886) and M. Fried and S. Unguru, *Apollonius of Perga's* Conics (Leiden: Brill, 2001).

[54] Apollonius used the term "hyperbola" to refer to just one branch of the modern curve and referred to a pair of branches as "opposite sections." Nevertheless, he realized there was an important relationship between the two and in many ways treated them under a unified approach.

of problems in the analytic corpus, and Apollonius is careful to point out how these books will be useful to various aspects of geometrical analysis. The goal of the treatise, then, was to furnish theorems useful to analysis in such a way that the theory of conic sections would be laid on a solid foundation, analogous to that of elementary geometry, and so that the topics which were raised would be handled with a certain completeness.

Apollonius' tendency to treat things exhaustively is also seen in the other area to which he made extensive contributions, the field of analysis. Besides the *Conics*, Apollonius wrote six of the treatises that Pappus mentions as belonging to the "field of analysis." Of these, *Cutting off a Ratio* survives in an Arabic translation,[55] and the others can be plausibly reconstructed based on Pappus' description, at least as far as the general structure and the methods used.[56] In contrast to Archimedes' short works, the topics covered in these treatises do not seem to be inherently interesting. For example, *Cutting off a Ratio* exhaustively solves the problem of producing a line through a given point cutting two given lines such that the segments cut off between the intersections and two given points on the lines have a given ratio.[57] The treatment of this mundane problem is then carried out in twenty-one cases for the arrangements of the original given objects (*dispositions*), and eighty-seven cases for the arrangements of the line that solves the problem (*occurrences*). The entire treatment is analytical, giving analyzed propositions for both problems and theorems, and stating the total number of possible solutions and specifying their limits. Although it is possible that some of this material could have been of use in studying conic sections, recalling that Apollonius claims that the *Conics* itself was to be of use in analysis, it seems likely that his short works were meant to be training texts in analytical methods. By reading these works, one could develop a strategy for formulating an analytical approach, as well as considerable experience in understanding the structure of analytical arguments.

When we look at the scope of his work, it appears that Apollonius viewed himself as Euclid's successor. He produced a body of texts that could be studied as a course in analysis, dividing up the subject matter of these treatises according to the types of constructions involved and crowned by what he rightly believed would become the elements of conic theory. Nevertheless, Apollonius, naturally, brought his own personal interests to his work. In contrast to Archimedes, he did not spend much effort on questions of mensuration but focused on relative placement and arrangement, treating at considerable length topics such as the number and location of the intersections of curved lines, the disposition of maximal and minimal

[55] R. Rashed and H. Bellosta, *Apollonius de Perge, La section des droits selon des rapports* (Berlin: Walter de Gruyter, 2010).
[56] For summaries of the contents of these lost treatises Jones, *Book 7 of the Collection*, pp. 510–619.
[57] For an overview of *Cutting off a Ratio*, see K. Saito and N. Sidoli, "The Function of Diorism in Ancient Greek Analysis," *Historia Mathematica* 37 (2010), 595–608.

lines, the placement and arrangement of tangents, and so forth. Moreover, he took Euclid's inclination to completeness to new heights and treated a number of topics so exhaustively as to strike many modern readers as tedious.

The Hellenistic period was the most active for researches in pure geometry. As well as the three authors we have surveyed, a number of other important mathematicians were at work during this period, most of whom are now known to us only by name. Nevertheless, in the works of these others which do survive in some form, such as Diocles' *Burning Mirrors*, in Arabic,[58] or Hypsicles' treatment of a dodecahedron and the icosahedron inscribed in the same sphere, modified to make an *Elements* XIV,[59] we find explicit mention of the work of Euclid, Archimedes, and Apollonius and a clear indication of respect. Despite everything that we have lost, it seems clear, from the regard in which they were held by their contemporaries and successors, that these three mathematicians were the most active geometers of the Hellenistic period and that we are able to make a fair appraisal of their efforts.

ARITHMETICS AND ALGEBRAIC THINKING

There has been much debate as to the role that algebra and algebraic thinking have played in Greek mathematics. Although few scholars still argue for interpreting the Greek theory of the application of areas as "geometric algebra,"[60] there is still considerable evidence for algebraic modes of thought in arithmetical problem solving and the theories devoted to this activity. Nevertheless, depending on how we define our terms, it is still possible to argue that Greek activity falls short of algebra. If, on the one hand, we regard algebra as an explicit study of equations and their methods of solution as reduced to the arithmetic operations, then it is possible to argue that Greek work in this area was proto-algebraic. If, on the other hand, we regard algebra as the use of an explicit unknown, the application of various stated operations to equations, and the exposition, through examples, of methods that have various applications, then Greek mathematicians did, indeed, do algebra. Whatever the case, for this style of mathematics we can use the term *arithmetics*, which is not far from the Greek usage.

[58] G. J. Toomer, *Diocles, On Burning Mirrors* (New York: Springer, 1976) and R. Rashed, *Les catoptriciens grecs* (Paris: Les Belles Lettres, 2002).

[59] B. Vitrac and A. Djebbar, "Le Livre XIV des Élements d'Euclid : versions greques et arabe (première partie, seconde partie)," *SCIAMVS* 12 (2011), 29–158, 13 (2012), 3–158.

[60] The subject of "geometric algebra" was at the core of a debate on the historiography of Greek mathematics that erupted in the 1970s, following a paper by S. Unguru. For a collection of these papers see, S. Unguru (ed.), "Methodological Issues in the Historiography of Greek Mathematics," in J. Christianidis (ed.), *Classics in the History of Greek Mathematics* (Dordrecht: Kluwer, 2004), pp. 383–461.

Reflecting the various subscientific traditions of practical, and recreational, mathematics, some examples of problems in practical arithmetics, of the kind that would have been taught in schools, have survived on fragments of papyri and excerpted in compilations, especially in the Heronian corpus.[61] In the case of elementary arithmetics, which are clearly the products of an oral tradition, no justification of the method of solution is given and no operational, or algebraic, reasoning is explicitly invoked. Understanding and reapplying the methods for which these sources are evidence would have required verbal explanations provided by a teacher. Although most of the problems in these traditions are of the first degree, there are a number of second-degree problems as well that appear to have been solved by the application of a set of identities that must have formed part of the oral instruction.[62] Learning to solve these sorts of problems was probably part of the education of professionals such as builders and accountants, and may have been part of the general education of the literate. Because second-degree equations were not used for practical applications before the early modern period, their presence in our early sources indicates that, even in this practical tradition, the development of general problem-solving skills was a goal of mathematical practice and education.

Concurrent with this practical tradition, Greek mathematicians produced an advanced, theoretical form of arithmetics that went well beyond the practical needs of schoolteachers and engineers. Our knowledge of higher Greek arithmetics is due almost entirely to the *Arithmetics* of Diophantus. Although there were probably other works in this tradition, and Diophantus himself refers to another of his own works, the *Porisms*, none of these have survived.[63] Hence, our only knowledge of this tradition comes through the rather idiosyncratic work of Diophantus, who, like the Hellenistic geometers, was more interested in solving problems than in giving a general exposition of the methods he employed.

All that we may say with certainty about Diophantus is that he was associated with Alexandria and lived sometime between the middle of the second century BCE and the middle of the fourth century CE. These dates derive from the fact that he mentions Hypsicles (ca. mid-second to early first century BCE) in his *On Polygonal Numbers*, and is, in turn, mentioned by Theon of Alexandria (ca. late fourth century CE).[64]

[61] For a discussion of the evidence from papyri for the practical tradition of Greek mathematics, which shows evidence of its connection with Mesopotamian and Egyptian sources, see J. Friberg, *Unexpected Links Between Egyptian and Babylonian Mathematics* (Singapore: World Scientific, 2005), pp. 105–268.

[62] For texts and translations of second-degree problems in elementary arithmetics from the Greek tradition, see J. Sesiano, "An Early Form of Greek Algebra," *Centaurus* 40 (1998), 276–302.

[63] J. Christianidis raises the possibility of a lost work called *Elements of Arithmetics*, but the evidence is not certain. See. J. Christianidis, "Ἀριθμητικὴ Στοιχείωσις: un traité perdu de Diophante d'Alexandrie?," *Historia Mathematica* 18 (1991), 239–46.

[64] P. Tannery, *Diophanti Alexandrini opera omnia*, 2 vols. (Leibzig: Teubner, 1893), vol. 1, 470; vol. 2, 35. See also F. Acerbi, *Diofanto, De Polygonis numeris* (Pisa: Fabrizio Serra Editore, 2011).

Our knowledge of Diophantus' most famous work, the *Arithmetics*, is based on both the Greek and Arabic traditions.[65] The Arabic version, based on a single manuscript, consists of four books that are not found in the Greek version. Moreover, there are a number of differences of presentation between the Greek and Arabic books. While the Greek text employs symbolic abbreviations that are explained in the text, the Arabic version is fully rhetorical, conforming to the practice in earlier medieval algebraic texts in Arabic. Furthermore, the Arabic text contains far more elementary details, going through the resolution of simple equations, giving full computations and verifying that the numbers so found solve the original problem. Hence, it has been suggested that the Arabic books are derived from an otherwise lost *Commentary to the Arithmetics* produced by Hypatia (ca. late fourth to early fifth century CE).[66]

For the historian of mathematics, the *Arithmetics* is a difficult text because it contains considerable mathematics but few general discussions of method. Hence, it is susceptible to a broad range of interpretations. Nevertheless, the discovery of the Arabic books has made it clear that Diophantus teaches his methodology in much the same way as Apollonius in *Cutting off a Ratio*, by repeated exposure to the application of specific methods. The *Arithmetics* provides a general framework for approaching problems and then sets out many different types of problems that can be handled by these methods.[67] The treatise is structured as a series of problems, where *problem* has the sense that it generally does in Greek mathematics. The focus of the treatment is more on the production of certain numbers that meet the conditions of the problem and less on the resolution of the equation that finds these numbers, which is often alluded to only briefly. Most of the text of a problem in the *Arithmetics* is devoted to setting out the problem in the terms of the "arithmetical theory," while the solution of the equation that results from this is relatively short and often referred to with a cryptic reference to the operations involved, such as "let the common wanting [terms] be added and

The testimonials that are used to date Diophantus more precisely, to the middle of the third century, date from the eleventh century. They can be, and have been, called into question. For example, see W. Knorr, "Arthimêtikê stoicheiôsis: On Diophantus and Hero of Alexandria," *Historia Mathematica* 20 (1933), 180–92, who also argues that Diophantus wrote a work on the elements of arithmetics.

[65] The Greek editions are Tannery, *Diophantus* and A. Allard, *Diophante d'Alexandrie, Les Arithmetiques* (Tourpes: Fonds national de la recherche scientifique, 1980). The Arabic editions are J. Sesiano, *Books IV to VII of Diophantus'* Arithemetica (New York: Springer, 1982), and R. Rashed, *Diophante, Les Arithmétiques* (Paris: Belles Lettres, 1984).

[66] This commentary is attributed to Hypatia in the *Suda*; see Tannery, *Diophantus*, vol. 2, 36. It should be noted that this attribution can be questioned.

[67] For general accounts of the methods used by Diophantus, see T. L. Heath, *Diophantus of Alexandria* (Cambridge: Cambridge University Press, 1964), pp. 54–98, Sesiano, *Books IV to VII of Diophantus' Arithemtica*, pp. 6–7, A. Bernard and J. Christianidis, "A New Analytical Framework for the Understanding of Diophantus' *Arithmetica* I–III," *Archive for History of Exact Sciences* 66 (2012), 1–69, and J. Christianidis and J. Oaks, "Practicing Algebra in Late Antiquity: The Problem-Solving of Diophantus of Alexandria," *Historia Mathematica*, 40 (2013), 127–63.

like [terms taken] from like."[68] This constructive aspect of the project is expressed by the refrain that concludes many of the propositions, in which Diophantus points out that the produced numbers "make the problem" (*poiousi to problēma*). As in geometric problems, the goal is to produce a specific object.

There are sometimes necessary conditions that must be stipulated to make a rational solution possible. The only times Diophantus explicitly refers to such a condition he uses the term *prosdiorismos*, but this may be because in these particular cases he is actually introducing a further condition, so that the word should be understood to mean "further specification." In a number of cases he uses *diorizesthai*, a verbalization of the standard *diorismos*, for initial specifications.[69] Diophantus is only concerned to give one solution, despite the fact that he is aware that there are often more possible solutions.

In his introduction to the work, Diophantus explains the basics of his method of applying symbolic abbreviations and the use of an unknown number. He gives a symbol for squares, δ^{υ}, cubes, κ^{υ}, and units, $\mathring{\mu}$, and states that a number that is some multitude of units is "called an unknown (*alogos*) number, and its sign is ϛ."[70] There is also a sign introducing wanting terms, ⋔, which always follows the extant terms.[71] With these symbols, what we call a polynomial can be written as a series of signs and numerals. For example, ϛ$\overline{\varepsilon}\mathring{\mu}\,\overline{\alpha}$ ⋔ $\delta^{\upsilon}\overline{\kappa\beta}$ can be anachronistically transcribed as $5x + 1 - 22x^2$. Diophantus, however, did not have the concept of a polynomial as made up of a number of terms related by operations, but rather expressed a collection of various numbers of different kinds (*eidos*) of things, some of which might be in deficit, where certain stated operations could be carried out on equations involving these sets of things.[72] Perhaps we should think of ϛ$\overline{\varepsilon}\mathring{\mu}\,\overline{\alpha}$ ⋔ $\delta^{\upsilon}\overline{\kappa\beta}$ as expressing something like "$5x$, 1, less $22x^2$."

Using these symbols, in *Arithmetics* I, as a means of demonstrating the utility of his new methods, Diophantus treats the sort of simple problems whose solutions would have been taught in many schools and would have been well known to professionals who used arithmetic methods. He shows how to use the conditions of the problem to produce an equation involving an unknown number, which can then be solved to produce the sought numbers, where all of the numbers involved are assumed to be rational. In *Arithmetics* II and III, Diophantus begins to treat indeterminate problems that are handled by finding an equation involving the unknown number,

[68] Tannery, *Diophantus*, p. 90. See J. Christianidis, "The Way of Diophantus: Some Clarifications on Diophantus' Method of Solution," *Historia Mathematica* 34 (2007), 289–305.

[69] Acerbi discusses these issues on pp. 10–11 of F. Acerbi, "The Meaning of πλασματικόν in Diophantus' Arithmetica," *Archive for History of Exact Sciences* 63 (2009), 5–31.

[70] Allard, *Diophante*, p. 375.

[71] Wanting terms are mathematically equivalent to negative terms; however, ancient and medieval mathematicians did not seem to think of them in terms of operations.

[72] Christianidis and Oaks, "Practicing Algebra in Late Antiquity."

which must be set equal to a square. A number of the problems solved in *Arithmetics* II, namely 8–11 and 19, compose a toolbox used extensively in the rest of the treatise. In the Arabic books, IV–VII, Diophantus continues to show how the methods set out in *Arithmetics* II can be applied to more difficult problems, now involving higher powers of the unknown. The introduction to *Arithmetics* IV introduces a third operation that allows us to reduce an equation containing two powers of the unknown to one with a power equal to a number, although this operation has already been used a few times in the previous books.[73] This allows the treatment of equations with higher powers of the unknown. In the last three Greek books, the problems become yet more difficult, often involving the construction of an auxiliary problem because the solution to the original problem requires special conditions to be met in order to make a rational solution possible. The final Greek book treats problems involving the determination of metric properties of right triangles, given various conditions.

Greek arithmetics exhibits a number of features that we regard as essential to algebra, such as the use of an unknown and the application of algebraic operations. Nevertheless, a number of other important features of algebra are not present. For example, the solutions and types of equations, as such, do not seem to have been a subject of direct study. Hence, arithmetics appears to have formed a group of problem-solving methods that were used as techniques for the production of sought numbers but which did not focus on the equation as a subject of study.

COMBINATORICS

Although it used to be believed that the Greeks did no substantial work in combinatorics, it is now clear that this assessment was simply due to a loss of the primary sources. The evidence that Greek mathematicians worked in combinatorics is slight, either due to the vagaries of our transmission or to a relatively narrow range of their activity. Although we find a number of recursive arguments and proofs in various mathematical works, and some clear combinatorial statements in later authors, such as Pappus, it is not clear that any works devoted to combinatorial mathematics were written. What seems much more likely is that combinatorial methods were developed in the context of technical works on logic and then disappeared with the loss of these texts.[74]

The most certain testimony comes from Plutarch, who tells us that all "the arithmeticians" and particularly Hipparchus (ca. mid-second century

[73] Sesiano, *Books IV to VII of Diophantus' Arithemtica*, p. 284.

[74] It has also been argued that Archimedes did work in combinatorics, but the actual evidence for this is rather slight. See R. Netz, F. Acerbi, and N. Wilson, "Towards a Reconstruction of Archimedes' *Stomachion*," *SCIAMVS* 5 (2004), 67–99, and G. Morelli, "Lo Stomachion di Archimede nelle testimonianze antiche," *Bollettino di Storia delle Scienze Matematiche* 29 (2009), 181–206.

BCE) contradict the claim made by Chrysippus that the number of conjunctions produced through ten assertibles is greater than a million.[75] In fact, Hipparchus calculated that the number of conjoined assertables for affirmation is 103, 049 while that for negation is 310, 954.[76] It has been shown that these numbers are the tenth Schröder number and half the sum of the tenth and the eleventh Schröder numbers, which are the correct solutions to a well-defined problem involving bracketing ten assertions, and their negations.[77] Since the calculation of these numbers could almost certainly not have been carried out by brute force, it is clear that Hipparchus must have had at his disposal a body of combinatorial techniques that he could draw on in formulating and solving this problem. Moreover, a survey of the ancient literature on logic reveals a concern with combinatorial thinking, which goes back at least as far as Aristotle.[78] It seems likely that combinatorial methods were developed in the technical study of logic, which naturally gave rise to many combinatorial assertions and problems.

THE EXACT SCIENCES

The Greek exact sciences were produced by people who regarded themselves, and were regarded by their contemporaries, as mathematicians. There was no special institutional setting for people who did optics or astronomy. What divided optics from geometry, or observational astronomy from spherics, was the different traditions of texts in which these fields were transmitted. Nevertheless, ancient thinkers divided up the disciplinary space of the mathematical sciences in various ways, which probably reflected individual interests and tastes as much as institutional or educational realities. Plato famously justified the Pythagorean division of the mathematical sciences into arithmetic, geometry, music, and astronomy – what would later be known as the quadrivium – but reading the work of his contemporary Archytas, for example, makes it clear that an evaluation of the contents and methods of these early sciences must not be made on the basis of later categories.[79] Aristotle placed the mathematical sciences as one branch of the theoretical sciences, between theology and physics;[80] and within the mathematical sciences he asserted various subordinate

[75] Plutarch, *On the Contradictions of the Stoics*, 1047c–e.

[76] The manuscripts actually have 310,952 for the second number, but this appears to be an error.

[77] R. P. Stanley, "Hipparchus, Plutarch, Schröder, and Hough," *The American Mathematical Monthly* 104 (1997), 344–350, and L. Habsieger, M. Kazarian, and S. Lando, "On the Second Number of Plutarch," *The American Mathematical Monthly* 105 (1998), 446.

[78] The ancient evidence on combinatorics has been surveyed by F. Acerbi, who gives a full account of Hipparchus' approach. See F. Acerbi, "On the Shoulders of Hipparchus: A Reappraisal of Ancient Greek Combinatorics," *Archive for History of Exact Sciences* 57 (2003), 465–502.

[79] Plato, *Republic*, VII 520a–532c, and Archytas, frags. 1 and 3.

[80] For example, Aristotle, *Metaphysics*, 6.1, 1026a and 11.7, 1064a–b.

relationships, such as optics to geometry, and harmonics to arithmetic.[81] Geminus (ca. first century BCE), in some general work on mathematics, provides an extensive classification of the mathematical sciences that reflects the variety of texts that we still possess. His first division was into pure and applied mathematics, of which the applied branches form what we call the exact sciences. The branches of pure mathematics were arithmetic and geometry, while those of applied mathematics were mechanics, astronomy, optics, geodesy, harmonics (*kanonikē*), and calculation (*logistikē*).[82] Ptolemy, who considered himself a mathematician, seems to agree with Aristotle's division of the theoretical sciences, but he took his point of departure by asserting that only mathematics is capable of producing knowledge and placing it, in this sense, above the other two.[83] Moreover, in the *Harmonics*, he makes it clear that he regards mathematics as the study of beautiful things, of which the highest branches are astronomy and harmonics.[84] In this way, geometry and arithmetic are reduced to the position of "indisputable instruments."[85] As these examples make clear, there was no generally accepted classification of the mathematical sciences and the organization and importance of the different fields was highly influenced by the classifier's own interests.

An important area of mathematics that was necessary in the exact sciences was the application of numerical methods to problems of mensuration. Although these techniques were developed by a number of mathematicians in the Hellenistic and Imperial periods, most of the surviving texts of this type are preserved under the authorship of Heron. While the Heronian texts also preserve many problems in the subscientific traditions that would have formed part of the education of professionals, such as engineers and accountants who used mathematics in their work, one of Heron's primary goals appears to have been to produce academic mechanics and theories of mensuration following the pattern of some of the more established exact sciences such as optics or astronomy.[86] Heron blends the methods of the Hellenistic geometers with the interests and approaches of mechanics and other practical fields, and, unsurprisingly, he mentions Archimedes more than any other single author. In Heron's work, we find a number of practices that typify the Greek exact sciences. He uses the structures of mathematical propositions, such as theorems, problems, analysis, and synthesis, although their content and actual meaning is altered to accommodate the new subject

81 R. D. McKirahan, "Aristotle's Subordinate Sciences," *British Journal for the History of Science* 11 (1978), 197–220.

82 Geminus' classification is quoted by Proclus. See Friedlein, *Procli Diadochi*, pp. 38–42. See also J. Evans and J. L. Berggren, *Geminos' Introduction to the Phenomena* (Princeton, NJ: Princeton University Press, 2006), pp. 43–8.

83 Ptolemy, *Almagest* I 1.

84 Ptolemy, *Harmonics* III 3.

85 I. Düring, *Die Harmonielehre des Klaudios Ptolemaios* (Gothenburg: Elanders, 1930) p. 94.

86 K. Tybjerg, "Hero of Alexandria's Mechanical Geometry," *Aperion* 37 (2004), 29–56.

matter. He models objects in the physical world using geometric diagrams and then focuses his discussion on these diagrams.[87] He blends metrical, computational approaches with geometric, constructive ones, and uses arguments involving *givens* to provide a theoretical justification for computations.[88] Although Heron is one of the first authors we have who uses these techniques, and although his field was mechanics and the mathematics of engineering and architecture, the fact that similar techniques are found in the astronomical writings of Ptolemy indicates that they were probably general methods of the exact sciences developed in the middle or end of the Hellenistic period.

In terms of the development of Greek mathematics, the most significant exact science was astronomy. For the purposes of handling problems that arose in astronomy, Greek mathematicians developed a number of mathematical techniques that are only found in astronomical texts but became an essential part of the canonical education of mathematicians in the late ancient and medieval periods.

For the purposes of ancient astronomy, timekeeping, and cosmology, Greek mathematicians developed a branch of applied geometry known as *spherics*. Spherics was the development of a theoretical geometry of the sphere and its application to problems of spherical astronomy, that is, the study of the motion of the fixed stars and the sun, regarded as located at some point on the ecliptic. Some works in spherics were probably produced in the classical period, by Eudoxus and others; however, as usual, the first texts in this field that survive are from the beginning of the Hellenistic period. We have works in pure spherical astronomy by Autolycus and Euclid, but it is clear from the way these works are presented that there was also a body of knowledge of spherical geometry on which they could draw.[89]

Towards the end of the Hellenistic period, Theodosius produced a new edition of elementary spherical geometry, his *Spherics*, which was so successful that previous versions of this material only maintained historical value and were eventually lost.[90] Theodosius' *Spherics* is in three books, the first of which is purely geometrical and the second two of which deal with topics applicable to spherical geometry, but still expressed in an almost purely geometrical idiom. The first book treats the properties of lesser circles and great circles of a sphere that are analogous with the properties of the chords and diameters of a circle in *Elements* III. The second book explores those properties of lesser circles and great circles of a sphere that are analogous with

87 See, for example, Heron, *Dioptra* 35.
88 See, for example, Heron, *Measurements* I 7 and 8; and *Measurements* I 10, or *Measurements* III.
89 J. Mogenet, *Autolycus de Pitane* (Louvain: Bibliothèque de l'Université, 1950) and Euclid, *Phaenomena et Scripta musica*, in Menge (ed.), *Euclidis Opera omnia*, vol. 5 (1916).
90 See J. L. Heiberg, *Theodosius Tripolites Sphaerica* (Berlin: Weidmann, 1927) and C. Czinczeheim, "Édition, traduction et commentaire des Sphériques de Théodose" (PhD thesis, Paris IV, 2000).

those of circles and lines in *Elements* III, which leads to theory of tangency, and theorems dealing with the relationships between great circles and sets of parallel lesser circles. Although still expressed in almost purely geometrical terms, the book ends with a number of theorems of largely astronomical interest, having to do with circles that can model the horizon, the equator, and the always visible, and always invisible, circles. The third book deals with what we would call the transformation of coordinates, or the projection of points of one great circle onto another, and concludes with theorems that can be interpreted as concerning the rising- and setting-times of arcs of the ecliptic, again without naming these objects explicitly, as Euclid had done, over 200 years earlier, in his *Phenomena*.[91]

Around the end of the first century CE, Menelaus took a new approach to spherical geometry that focused on the geometry of great circles on a sphere and was orientated towards Hellenistic developments in trigonometry, enabling him to produce the theoretical basis for a more elegant, metrical spherical astronomy. Only fragments of the Greek text of Menelaus' *Spherics* survive,[92] but we have a number of Arabic, Latin, and Hebrew versions. Our knowledge of this work is somewhat tentative because only late, heavily modified, versions of the text have been edited or studied in depth; nevertheless, we may sketch out the general approach and trajectory of the treatise.[93] Book I develops a theory of the congruency of spherical triangles modeled on the first part of *Elements* I, but including the congruency of pairs of triangles having three equal angles.[94] This is then followed by a run of theorems that involve inequalities that can be asserted in a given spherical triangle in which the sum of two sides is equal to, greater than, or less than a semicircle. Book II, then, develops a theory of bundles of lesser circles having some angle with a great circle, acting as an analogy with the theory of parallels in *Elements* I. The versatility of this approach is then demonstrated with three theorems on the intersections of great circles with sets of parallel lesser circles, which elegantly do the work of six long theorems in Theodosius' *Spherics* III. Book III begins with the theorem passed down under Menelaus' name, but which may have been already well-known in his time,[95] which gives a compound ratio between the chords that subtend various parts of a convex quadrilateral made up of great circles. This is

91 For an overview of Theodosius' approach see J. L. Berggren, "The Relation of Greek Spherics to Early Greek Astronomy," in A. Bowen (ed.), *Science and Philosophy in Classical Greece* (New York: Garland, 1991), pp. 227–48. The text of the *Phenomena* is in Menge (ed.), *Euclidis Opera omnia*, vol. 5. There is an English translation by J. L. Berggren and R. S. D. Thomas: *Euclid's Phaenomena: A Translation and Study of a Hellenistic Treatise in Spherical Astronomy* (New York: Garland, 1996).

92 The Greek fragments are collected on pages 22–5 of A. Björnbo, "Studien über Menelaos' *Sphärik*," *Abhandlungen zur Geschichte der mathematischen Wissenshaften* 14 (1902).

93 For Ibn 'Irāq's version, see M. Krause, *Die Spärik von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. 'Alī b. 'Irāq* (Berlin: Weidmann, 1936).

94 Not even the book divisions are consistent in the various versions of the text; see Krause, *Die Spärik von Menelaos*, pp. 8–9. The following numbers are those in Krause's edition.

95 N. Sidoli, "The Sector Theorem Attributed to Menelaus," *SCIAMVS* 7 (2006), 43–79.

then combined with Menelaus' theory of spherical triangles and his theory of bundles of parallel lesser circles to provide tools for developing a fully metrical spherical astronomy.[96] In this way, Menelaus combined the Hellenistic trigonometric methods with his new theory of the spherical triangle to produce a new approach to spherical trigonometry.

A major area of mathematical development in astronomical texts was trigonometry, which in antiquity always literally involved the mensuration of elements of a triangle. We may talk of Greek trigonometry in three main stages.[97] The first stage, which is proto-trigonometric, is attested in the astronomical writings of Aristarchus and Archimedes, *On the Sizes and Distances of the Sun and the Moon* and the *Sand Reckoner*, respectively. In these texts, the authors implicitly rely on a pair of ratio inequalities that hold between a ratio of corresponding sides and a ratio of corresponding angles of a pair of triangles. Both Aristarchus and Archimedes seem to assume these inequalities as well-known lemmas, but we have a number of proofs of their validity in later authors.[98] Using these inequalities, Greek mathematicians were able to derive fairly precise upper and lower bounds on sought angles or lengths. There were, however, a number of difficulties involved in this method. The computations were cumbersome, and the more lengthy the computation, the more accuracy would be lost in regard to the term to be bounded. Furthermore, these inequalities are only sufficiently precise for small angles, and if large angles were involved they would produce substantially inaccurate results. Although these methods were sufficient for the laborious calculation of a few chosen values it is difficult to imagine that an accurate, predictive astronomy could have been established using such techniques.

The next stage in the development of Greek trigonometry, although not well documented in our sources, appears to have been directly spurred by the desire to produce a predictive astronomy based on geometric models. During the Hellenistic period, Greek astronomers came into possession of Babylonian sources that showed them that it was possible to produce an accurate, predictive astronomy, and provided them with observation reports and numerical parameters to put this project into effect.[99] By at least the time of Hipparchus, in the second century BCE, the goal was to produce a numerically predictive astronomy on the basis of geometric models. In order to meet this goal, Greek

[96] R. Nadal, A. Taha, and P. Pinel, "Le contenu astronomique des *Sphériques* de Ménélaos," *Archive of History of Exact Sciences* 58 (2004), 381–436.

[97] For an overview of the history of Greek trigonometry, see G. Van Brummelen, *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry* (Princeton, NJ: Princeton University Press, 2009), pp. 20–90.

[98] W. Knorr, "Ancient Versions of Two Trigonometric Lemmas," *Classical Quarterly* 35 (1985), 361–91.

[99] A. Jones, "The Adaptation of Babylonian Methods in Greek Numerical Astronomy," *Isis* 82 (1991), 441–53, and A. Jones, "On Babylonian Astronomy and its Greek Metamorphoses," in F. J. Ragep and S. Ragep (eds.), *Tradition, Transmission, Transformation* (Leiden: Brill, 1996) pp. 139–55.

mathematicians developed tabular trigonometry, which consisted of a group of theorems about right triangles and tables that allowed a length to be determined as a function of a given angle, and conversely. Tabular trigonometry allowed them to use observed angles and period relations to calculate the numerical parameters of assumed geometric models. Although it is not certain that Hipparchus was the first to produce a chord table, there is evidence that Hipparchus tabulated the chords subtending angles at $7\frac{1}{2}^\circ$ intervals, from 0° to 180°, for a circle with $d = 6{,}875$.[100] Using such chord tables, mathematicians, such as Hipparchus, Diodorus (ca. mid-first century BCE), and Menelaus, were able to solve computational problems in plane and spherical trigonometry by treating them with forms that we would write using the sine function, since $\mathrm{Crd}(2\alpha) = 2\,\sin\alpha$.

The final stage of Greek trigonometry is found in the writings of Claudius Ptolemy. In his *Almagest* I 10–13, Ptolemy sets out the mathematics necessary for the trigonometric methods he will use. He begins by showing how a chord table of $1/2^\circ$ intervals, on a circle with $d = 120$, could be calculated using a number of geometrically derived formulas and an approximation of the chord of $1/2^\circ$ using a lemma similar to that at the basis of the proto-trigonometric tradition. This serves as a justification for his more precise chord table, although Ptolemy does not actually say that he used these methods to derive it.[101] This more precise chord table was the computational tool at the foundation of Ptolemy's plane and spherical trigonometry, as found in works such as the *Almagest*, the *Planisphere*, and the *Analemma*.[102] For spherical trigonometry, Ptolemy restricted himself to the theorem based on a convex quadrilateral of great circle arcs, known as the Menelaus theorem, avoiding any use of the more powerful spherical-trigonometric methods that Menelaus had developed.

A final mathematical tool of the exact sciences should be mentioned: numerical tables. Tables presumably entered Greek mathematical practice along with predictive astronomy from Babylonian sources during the Hellenistic period. Although mathematicians such as Hipparchus, Diodorus, and Menelaus must have used tables, again, our earliest texts that contain tables are by Ptolemy, particularly his *Almagest, Analemma*, and *Handy Tables*. Despite the fact that tables were used long before the function

100 The argument for the details of Hipparchus' chord table is given in G. Toomer, "The Chord Table of Hipparchus and the Early History of Greek Trigonometry," *Centaurus* 18 (1973), 6–28, and D. Duke, "Hipparchus' Eclipse Trios and Early Trigonometry," *Archive for History of Exact Science* 56 (2005), 427–33.

101 For an argument that Ptolemy actually used interpolation methods in constructing his chord table see G. Van Brummelen, "Mathematical Tables in Ptolemy's Almagest" (PhD thesis, Simon Fraser University, 1993), pp. 46–71.

102 J. L. Heiberg (ed.), *Claudii Ptolemaei opera omnia*, 2 vols. (Leipzig: Teubner, 1898–1903), N. Sidoli and J. L. Berggren, "The Arabic Version of Ptolemy's *Planisphere*," *SCIAMVS* 8 (2007), 37–139, and D. R. Edwards, "Ptolemy's Περί ἀναλήμματος" (PhD thesis, Brown University, 1984).

concept became explicit, they exhibit relationships that are similar to certain modern conceptions of a function.[103] For example, tables were treated as general relations between members of two, or more, sets of numbers. Moreover, the algorithms that describe how to use the tables make it clear that they describe a computational rule that maps a single member of the domain to a single member of the codomain. These proto-functions were used in a number of ways to handle the metrical aspects of geometrical objects and moving components of a geometrical model.

In Ptolemy's writings, tables are sets of numerical values that correspond to lengths and arcs in the geometric models from which they are derived. At least in principle, they are produced by direct derivation from geometric objects with assumed numeric values. We can understand the tables themselves as a numerical representation of the underlying model, which is geometric. The tables are then used, either by Ptolemy or by the reader, to provide an evaluation of specific numerical values that represent the underlying model. In the *Almagest*, mathematical tables are a component of Ptolemy's goal of producing a deductively organized description of the cosmos, presented in an essentially single argument.[104] Indeed, Ptolemy makes a number of explicit assertions that the structure of the tables in the *Almagest* should exhibit both the true nature of the phenomena in question and have a suitable correspondence with the mathematical models.[105] For Ptolemy, a table, like a mathematical theorem, is both a presentation of acquired mathematical knowledge and a tool for producing new mathematical results.

EXPOSITORS AND COMMENTATORS

Our knowledge of the substantial texts of Greek mathematics comes through the filter of the scholarship of the mathematicians of late antiquity, most of whom were associated with schools of philosophy and regarded mathematics as an important part of a broader cultural and educational project. We have seen that the texts of the earlier periods were edited and commented upon by these mathematical scholars, and this process acted as an informal process of selection, in so far as texts which did not receive attention had a dramatically reduced chance of being passed down.

These late-ancient scholars were primarily responsible for creating the image of theoretical mathematics that was transmitted to the various cultures around the Mediterranean in the medieval and early modern periods.

103 O. Pedersen, "Logistics and the Theory of Functions," *Archives Internationales d'Histoire des Sciences* 24 (1974), 29–50.
104 N. Sidoli, "Mathematical Tables in Ptolemy's Almagest," *Historia Mathematica* 41 (2014), 13–37.
105 See, for example, Heiberg (ed.), Claudii Ptolemaei opera omnia, vol. 1, 208, 251.

Through their teaching and scholarship, they established various canons of the great works of the past, arranged courses of study through select topics, reinforced a sound and lasting architecture by shoring up arguments and making justifications explicit, and, finally, secured their place in this tradition by intermingling their work with that of their predecessors and situating the whole project in contemporary modes of philosophic discourse.

One of the most impressive of these scholars was Pappus of Alexandria, who was a competent mathematician and a gifted teacher, who made important strides to associate mathematics with areas of interest in philosophy by constantly arguing for the relevance of mathematics to other aspects of intellectual life. Pappus worked in many areas of the exact sciences, wrote commentaries on canonical works, such as the *Elements* and the *Almagest*,[106] and produced a series of short studies that were later gathered together into the *Mathematical Collection*.[107] It is clear from Pappus' writing, that he was part of an extended community of mathematicians and students who had regard for his work and interest in his teaching.

Pappus' *Collection*, although incomplete, is indispensable for our understanding of both the early history of Greek mathematics and the main trends of mathematical thought in late antiquity.[108] Book II, which is fragmentary, discusses a system of large numbers attributed to Apollonius. Books III and IV present a number of topics in advanced geometry framed in philosophical discussions that show, among other things, how mathematical argument can be of relevance to philosophical issues.[109] Here we find discussions of the difference between theorems and problems; the division of geometry into linear, solid, and curvilinear methods; treatments of mechanical and geometrical curves; examples of problem solving through analysis and synthesis; discussions of geometrical paradoxes; and examples of solutions to three classical problems: squaring a circle; duplicating a cube; and trisecting an angle. Pappus' style is to situate his own work in a wealth of material drawn from historically famous mathematicians, to mix geometrical and mechanical approaches, and to combine numerical examples with pure geometry. Book V deals with isoperimetric and isovolumetric figures, again situating the discussion in the context of past work, for example, that of

[106] His commentary on *Elements* X survives in an Arabic translation; see W. Thomson and G. Junge, *The Commentary of Pappus on Book X of Euclid's Elements* (Cambridge, MA: Harvard University Press, 1930). What survives of his commentary on the *Almagest* was edited by A. Rome (ed.), *Commentaires de Pappus et Théon d'Alexandrie sur l'Almageste*, 3 vols. (Rome: Biblioteca Apostolica Vaticana, 1931–43), vol. 1: *Pappus, Commentaire sur les livres 5 et 6* (1931).

[107] Hultsch, *Pappi Alexandrini Collectionis*, Jones, *Book 7 of the Collection*, and H. Sefrin-Weis, *Pappus of Alexandria, Book 4 of the Collection* (London: Springer, 2010).

[108] For a study of Pappus in his intellectual context, see S. Cuomo, *Pappus of Alexandria and the Mathematics of Late Antiquity* (Cambridge: Cambridge University Press, 2000).

[109] A. Bernard, "Sophistic Aspects of Pappus' *Collection*," *Archive for History of Exact Sciences* 57 (2003), 93–150.

Archimedes, Zenodorus (ca. third to second century BCE), and Theodosius.[110] The next two books were clearly produced as part of Pappus' teaching activities, and they are mainly made up of lemmas to canonical works. Book VI deals with what Pappus calls "the field of astronomy," which came to be known as the *Little Astronomy*.[111] It organizes and discusses the treatises that students should master before studying Ptolemy's *Almagest*.[112] Book VII covers "the field of analysis," which consisted of a group of works by authors such as Aristaeus, Euclid, and Apollonius.[113] It begins with a general account of analysis, discusses the overall content of each work, and provides numerous lemmas for individual propositions, many of which give justifications for common practices of the Hellenistic geometers, such as operations on ratios and the use of compound ratios. Book VIII treats theoretical mechanics in much the same vein as Heron, whom Pappus refers to a number of times. It models various machines geometrically and sets out the mathematical theory of certain practical constructions.[114]

The other mathematical scholars of the late ancient period were also involved in teaching and expounding the classics, and hence mostly worked through the medium of commentaries. Theon of Alexandria, in the fourth century, edited works by Euclid and wrote commentaries to Ptolemy's *Almagest* and *Handy Tables*.[115] Hypatia, his daughter, collaborated with her father on various projects and wrote commentaries to Apollonius and Diophantus.[116] In the following century, Proclus of Athens wrote a commentary on the *Elements*. Eutocius of Ascalon, in the sixth century, edited works by Archimedes and Apollonius, and wrote commentaries to them.

This work was a continuation of a tradition of commentating and editing that began in the Imperial period. The scholars of this later period paid particular attention to issues of logical completeness, formal structure, and

[110] Another important text for this material is the anonymous introduction to Ptolemy's *Almagest*. See pp. 92–196 of F. Acerbi, N. Vinel, and B. Vitrac, "Les *Prolégomènes à l'Almageste*, Introduction générale – Parties I–III," *SCIAMVS* 11 (2010), 53–210.

[111] Hultsch, *Pappi Alexandrini Collectionis*, p. 474.

[112] For Pappus' treatment of Theodosius' *Spherics*, see M. Malpangotto, "Sul commento di Pappo d'Alessandria alle 'Sferiche' di Teodosio," *Bollettino di Storia delle Scienze Matematiche* 23 (2003), 121–48.

[113] Jones, *Book 7 of the Collection*, p. 83.

[114] The Arabic version of Book VIII, which has not yet been thoroughly studied, contains a number of topics not found in the Greek version. For an example, see D. E. P Jackson, "Towards a Resolution of the Problem of τὰ ἑνὶ διαστήματι γραφόμενα in Pappus' Collection Book VIII," *The Classical Quarterly* 30 (1980), 523–33.

[115] Rome (ed.), *Commentaires de Pappus et Théon d'Alexandrie sur l'Almageste*, J. Mogenet and A. Tihon, *Le "Grand Commentaire" de Théon d'Alexandrie aux Tables Faciles de Ptolémée, livre I* (Rome: BAV, 1985), A. Tihon, *Le "Grand Commentaire" de Théon d'Alexandrie aux Tables Faciles de Ptolémée, livres II, III*, 2 vols. (Rome: BAV, 1991–9), and A. Tihon, *Le "Petit Commentaire" de Théon d'Alexandrie aux Tables Faciles de Ptolémée* (Rome: BAV, 1979).

[116] For Hypatia's life and work, see M. Dzielska, *Hypatia of Alexandria* (Cambridge, MA: Harvard University Press, 1995).

readability. They produced fuller texts with more explicit arguments; wrote auxiliary lemmas; introduced internal references to other parts of the canon; restructured the treatises and individual elements of the text; added introductions and conclusions; advocated explicit classifications; rewrote theories from new perspectives; and summarized long works for the purposes of study.[117]

All of this was part of a broad trend, begun in the Imperial period by authors such as Geminus, Heron, and Ptolemy, to incorporate the mathematical sciences into the philosophical tradition.[118] Although in the Classical and early Hellenistic periods, philosophers showed interest in mathematical approaches, there is little indication that mathematicians had a similar regard for philosophy. The mathematicians of the late ancient period, however, were concerned that mathematics be part of an education in philosophy and rhetoric.[119] Their texts show a combination of modes of thought from the traditions of pure mathematics with those from the various exact sciences, and a mixture of philosophical concerns with mathematical issues. Their project, situated as it was in the philosophical schools, argued both explicitly and implicitly for the value of mathematics to philosophy. The final stages of Greek mathematical practice furnished the image of Greek mathematics that remains with us to this day: that of a fully explicit, interconnected, literary product.

CONCLUSION

Theoretical Greek mathematics underwent considerable change and development. What began as a leisure activity for elite scholars was then applied in solving interesting problems in the natural sciences, and it became of practical use to a wide range of scholars and professionals. The resulting combination of the theoretical and practical traditions was then institutionalized in courses in mathematical sciences at the philosophical schools. Throughout this long transition, however, Greek mathematics, as a type of intellectual activity, was primarily defined by the cohesion of certain traditions of practices and texts, not by the social position, or institutional setting, of the practitioners themselves.

[117] R. Netz, "Deuteronomic Texts: Late Antiquity and the History of Mathematics," *Revue d'histoire des mathématiques* 4 (1998), 261–88.

[118] For a discussion of the philosophy of Ptolemy, see L. Taub, *Ptolemy's Universe* (Chicago, IL: Open Court, 1993), and J. Feke and A. Jones, "Ptolemy," in L. P. Gerson (ed.), *The Cambridge History of Philosophy in Late Antiquity* (Cambridge: Cambridge University Press, 2010), pp. 197–209.

[119] A. Bernard, "Comment definer la nature des textes mathématiques de l'antiquité grecque tardive? Proposition de réforme de la notion de 'textes deutéronomiques'," *Revue d'histoire des mathématiques* 9 (2003), 131–73.