Show-Me Collection of Series

Contents

Introduction

This Collection is intended to be accessible to a wide audience with not only the formulae, but also the method by which each formula can be derived.

The 3-bar equals sign, $\equiv$, is used here to show a definition, and can be read as “is identically equal to”.

The *big sigma* notation uses a *dummy variable* at the bottom (in the example below, r) with the starting value (= 1), and the limiting value at the top, in this case n.

$$\sum_{r=1}^{n} r \equiv 1 + 2 + 3 + 4 + \ldots + n$$

The book ***Summation of Series***, collected by L.B.W. Jolley (2nd ed, 1961) has over 1000 series listed, but for each one you have to then look it up in one of 30 old books (many now 100 years old) to find out how it was derived. Here we list just a few since the derivations typically each take at least one page. (A few are referenced here as Jxxx). The book ***A Short Table of Integrals***, Peirce (Foster) 4th ed, also has a few series referenced as Pxxx).

The series here are labelled with a 3 digit number and optional post-fixed letter codes:

- **A** for *alternating* series, which means some terms are negative.
- **E** for an *example* using particular values.
- **F** for *finite*, typically a sum to n terms.
- **U** for *unbounded*, a sum to infinity. (**F** and **U** are mutually exclusive, and non-optional).

This collection is not intended to be comprehensive. The idea is that the methods and examples presented will allow simple non-tabulated series to be readily evaluated.

It has to be said that the tabulation of large lists of series is much less important in the modern world due to the existence of mathematical software which trivially calculates the sum of arbitrary series. Sadly such software is now so expensive that it is not necessarily readily available to all researchers. Also the methods used to solve the simple series presented here do not seem to be well represented in modern text books.

$\ln(x)$ is the natural logarithm of x. Higher mathematics text books apparently write this as $\log(x)$, with the base, e, being ‘understood’.

e = 2.7182818… = $\exp(1)$

Accuracy of such a collection of formulae is important, so all the formula with a box-border were numerically validated using Mathcad 15. The Mathcad file has also been made publically available.

This paper was written using MS Word 2003. This was a *total pain* since Word often corrupts the equations, leaving an uneditable image. Some equations had to be re-entered several times. However, newer versions of Word which don’t use Microsoft Equation Editor 3.1 do not seem to have the same capability to use clickable images as contents entries. The more recent versions of MathType are expensive and may not work as indexable lists.

The known faults meant it was essential to both save ***and close*** the file often, so that when an inevitable corruption occurred, the damage was limited. Just saving (without closing) Word is not a safe method of protecting your work.

Summation Methods

There are only a few different summation methods, and they are summarised here for completeness. You could think of these as little ‘tricks’, but they are so important that they are classified as standard methods.

Some non-obvious definitions may be useful:

$$1! \equiv 1 \qquad 0! \equiv 1$$

$$k^1 \equiv k \qquad k^0 \equiv 1$$

Infinite series can only be manipulated by the methods summarised below when they are convergent. Non-convergent (divergent) series require different methods.

It should be noted that there are really two different sources of series. The first type comes from some problem situation, and by evaluating the series we solve the problem. The second type is not necessarily distinguishable from the first type by looking at the series terms, but the key difference is that it is not possible to independently evaluate it.

This second type of series includes those which sum to some multiple of π, some power of the exponential function, or some logarithmic value. The point is that if somebody brings out a series and asks you to sum it, those which have a sum as a multiple of π (for example) will not be directly summable. You can of course compare this new series to known series using a handbook such as Jolley’s collection. This is much like the situation with calculus. Some integrals are evaluated indirectly by ‘guessing’ what function would need to be differentiated to get that value. Typically this second type of series is found from a Taylor series expansion of a function so the result is known, and the power series gives a way of evaluating the function.

Simplification

In the first instance one should inspect any new series to see if it can be split up into simpler pieces, each of which is individually tabulated in a collection of solutions. For example:

$$S_n = 3(1)+3(3)+3(5)+3(7)+3(9)+\ldots+3(2n-1)$$

is immediately reducible to $S_n = 3\left(1+3+5+7+9+\ldots+(2n-1)\right)$.

In other words we first remove additive constants and constant multipliers.

Simplification using *partial fraction* expansion is also useful in some cases.

$$\sum \frac{1}{r(r+1)} = \left(\sum \frac{1}{r}\right) - \left(\sum \frac{1}{r+1}\right)$$

The brackets on the right-hand-side can be useful, especially in collections, as it is sometimes not obvious when the big-sigma argument ends.

Series Reversal

This method only applies to ***arithmetic series***, that is series which have a common difference between terms.

$$S_n = 1 + 2 + 3 + \ldots + (n-4) + (n-3) + (n-2) + (n-1) + n$$

$$S_n = n + (n-1) + (n-2) + \ldots + 5 + 4 + 3 + 2 + 1$$

The same series is written twice, once in the normal increasing order, and then again in decreasing order. The terms are aligned ‘vertically’. The value “1” on the upper line aligns with the value “n” on the lower line. It is then clear that summing the two series in this vertically-aligned manner gives n pairs, each of which sum to n +1.

We can then immediately say $2S_n = n(n+1)$

This is a very simple and powerful method, but it has limited applicability.

Series Shifting

This method applies very generally and not only to ***geometric series***. A common factor between either the numerator or the denominator can be exploited to shift a series.

$$\begin{aligned} S &= 1 + k + k^2 + k^3 + k^4 + k^5 + k^6 + \dots \\ k \cdot S &= \phantom{1+{}} k + k^2 + k^3 + k^4 + k^5 + k^6 + \dots \end{aligned}$$

Having written S, we then multiply by the common multiplier to shift the series along one position. Similar terms can then be lined-up vertically and we can subtract $k \cdot S$ from S to eliminate all but one (or two) values. Note that there is no requirement here for k to be positive or an integer.

If $|k| < 1$ then the left-over part becomes vanishingly small as the number of terms increases without limit.

If $|k| > 1$ then the series does not have a finite sum. In this case you should do calculations based on the finite sum. When the two series are subtracted there will be a "left-over" part at the end, as well as at the beginning, which is why you get two terms remaining on subtraction.

Numerator Splitting to simplify the series

If the numerator doesn't quite fit to an existing pattern, we can split it to get two series that do fit some existing pattern.

$$S = 1 + \frac{2}{k} + \frac{3}{k^2} + \frac{4}{k^3} + \frac{5}{k^4} + \frac{6}{k^5} + \dots$$

$$S = 1 + \frac{1+1}{k} + \frac{1+2}{k^2} + \frac{1+3}{k^3} + \frac{1+4}{k^4} + \frac{1+5}{k^5} + \dots$$

$$S = 1 + \left(\sum_{r=1}^{\infty} \frac{1}{k^r} \right) + \left(\sum_{r=1}^{\infty} \frac{r}{k^r} \right)$$

Difference of Squares

First we shift, and then vertically re-align the terms:

$$S_n = \frac{1^2}{k} + \frac{2^2}{k^2} + \frac{3^2}{k^3} + \frac{4^2}{k^4} + \dots + \frac{n^2}{k^n}$$

$$\frac{S_n}{k} = \qquad \frac{1^2}{k^2} + \frac{2^2}{k^3} + \frac{3^2}{k^4} + \frac{4^2}{k^5} + \dots + \frac{n^2}{k^{n+1}}$$

$$S_n - \frac{S_n}{k} = \frac{1^2}{k} + \frac{2^2 - 1^2}{k^2} + \frac{3^2 - 2^2}{k^3} + \frac{4^2 - 3^2}{k^4} + \dots + \frac{n^2 - (n-1)^2}{k^n} - \frac{n^2}{k^{n+1}}$$

Looking at the general term:

$$n^2 - (n-1)^2 = n^2 - \left(n^2 - 2n + 1\right) = 2n - 1$$

So we now have two simpler series to deal with.

Convergence / Divergence

A sum to *n*-terms is ordinarily a definite thing, having a definite value. An infinite sum is not in the same category. Having found a sum to the first *n*-terms, it is important to understand that what happens as *n* increases without limit determines if the series has a definite value.

As an example, the sum of *all* the natural numbers does not have a finite value.

$$\sum_{r=1}^{\infty} r = \infty$$ ✘

We regard the summation statement above as invalid. The infinity symbol is a concept, not a value, so it is *improper* (wrong) to use an equals sign. Typically an arrow is used in place of an equals sign, as it *is* acceptable to say that numerical values "tend to infinity".

$$\sum_{r=1}^{\infty} r \to \infty$$

We call such a series *strictly divergent*. The partial sums (the sum of the first *n*-terms) form a strictly increasing sequence with *n*.

Infinite series either converge to a definite limit or they do not. Those which converge to a definite limit are known as *convergent series*. All the others are *divergent series*.

It is not certain that a given convergent series will converge to its limit with a strictly decreasing difference from the limit. *Monotonic convergence* is where the difference from the limit never increases. *Strict convergence* is a stronger statement in that the difference from the limit *always reduces* for each new term.

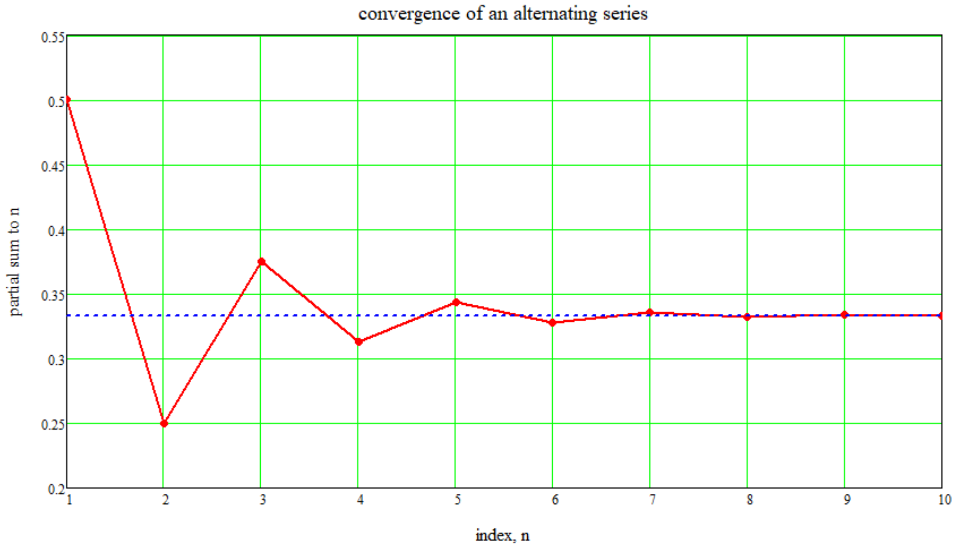
The geometric series below is strictly convergent, as seen by the partial sum formula. The infinite limit is of course 1. The limit is approached from below.

$$\sum_{r=1}^{n} \frac{1}{2^r} \equiv \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \ldots + \frac{1}{2^n} = \quad 1 - \frac{1}{2^n}$$

For the *alternating convergent series* below, the convergence occurs in a damped oscillatory manner.

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{2^r} \equiv \frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} + \ldots - \frac{(-1)^n}{2^n} = \quad \frac{1}{3}\left(1 - \frac{(-1)^n}{2^n}\right)$$

The limit is approached from above and below.



Testing for convergence is redundant when the partial sum can be expressed as a closed-form expression, as we have done above. However, such a closed-form sum may not always be achievable.

010F $\boxed{1+2+3+4+\ldots+n = \tfrac{1}{2}n(n+1)}$

Write the *arithmetic series* twice, one above the other, the second one running backwards:

$$\sum_{r=1}^{n} r \equiv 1 + 2 + 3 + 4 + \ldots + n$$

$$\sum_{r=1}^{n} r \equiv n + (n-1) + (n-2) + (n-3) + \ldots + 1$$

Summing vertically-aligned pairs we have *n* lots of (*n*+1).

$$2 \times \sum_{r=1}^{n} r = n(n+1)$$

$$\therefore \sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

010FE $\boxed{1+2+3+4+\ldots+n \cong \frac{n^2}{2} \quad \text{for large } n}$

As is evident from 010F.

010AFE $\boxed{1-2+3-4+\ldots-(-1)^n n \cong \frac{n}{2}(-1)^{n+1} \quad \text{for large } n}$

$$S_n \equiv \sum_{r=1}^{n} r\cdot(-1)^{r+1} \equiv 1-2+3-4+5-6+\ldots-(-1)^r n$$

If S_n is odd then:

$$S_n = 1+(-2+3)+(-4+5)+(-6+7)+\ldots[-(n-1)+n]$$

$$= 1+\frac{n-1}{2} \quad = \frac{n+1}{2}$$

If S_n is even then:

$$S_n = (1-2)+(3-4)+(5-6)+(7-8)+\ldots((n-1)-n)$$

$$= -\frac{n}{2}$$

Clearly this series has an oscillatory divergence.

010F $\boxed{(a+d)+(a+2d)+(a+3d)+\ldots+(a+nd)= \quad na+\tfrac{1}{2}nd\left(n+1\right)}$

A more general arithmetic series:

$$\sum_{r=1}^{n}(a+d\times r)\equiv(a+d)+(a+2d)+(a+3d)+\ldots+(a+nd)$$

Using the distributive rule for summation:

$$\sum_{r=1}^{n}(a+d\times r)=\left(\sum_{r=1}^{n}a\right) \quad + \quad d\times\left(\sum_{r=1}^{n}r\right)$$

$$\sum_{r=1}^{n}a \quad =na$$

$$d\times\sum_{r=1}^{n}r=d\times\tfrac{1}{2}n\left(n+1\right)$$

$$\therefore\sum_{r=1}^{n}(a+d\times r)=na+\tfrac{1}{2}nd\left(n+1\right)$$

$$\sum_{r=0}^{n-1}(a+d\times r)\equiv a+(a+d)+(a+2d)+(a+3d)+\ldots+(a+\left(n-1\right)d)$$

012F $\boxed{a+(a+d)+(a+2d)+(a+3d)+\ldots+(a+\left(n-1\right)d)= \quad na+\tfrac{1}{2}nd\left(n-1\right)}$

030F $\boxed{1+3+5+7+\ldots+\left(2n-1\right)= \quad n^2}$

The sum of the first *n* odd numbers:

$$\sum_{r=1}^{n}\left(2r-1\right) \quad \equiv \quad 1+3+5+7+\ldots+\left(2n-1\right)$$

$$\sum_{r=1}^{n}\left(2r-1\right) \quad =2\times\left(\sum_{r=1}^{n}r\right)-\sum_{r=1}^{n}1$$

$$2\times\sum_{r=1}^{n}r= \quad n\left(n+1\right)$$

$$\sum_{r=1}^{n}1= \quad n$$

$$\therefore\sum_{r=1}^{n}\left(2r-1\right) \quad = \quad n\left(n+1\right)-n \quad = \quad n^2$$

040F $\boxed{2+4+6+8+\ldots+2n = \quad n(n+1)}$

The sum of the first *n* even numbers:

$$\sum_{r=1}^{n} 2r \quad \equiv \quad 2+4+6+8+\ldots+2n$$

$$\therefore \sum_{r=1}^{n} 2r \quad = \quad 2\times\sum_{r=1}^{n} r \quad = \quad n(n+1)$$

050F $\boxed{k+k^2+k^3+k^4+\ldots+k^n = \quad \frac{k}{1-k}\left(1-k^n\right)}$

Geometric series:

$$S_n \equiv \sum_{r=1}^{n} k^r \equiv \quad k+k^2+k^3+k^4+\ldots+k^n$$

$$S_n = \quad k+k^2+k^3+k^4+\ldots+k^n$$

$$k\cdot S_n = \qquad k^2+k^3+k^4+k^5+\ldots+k^{n+1}$$

$$S_n - k\cdot S_n = k - k^{n+1}$$

$$(1-k)\cdot S_n = \quad k\left(1-k^n\right)$$

$$\therefore \sum_{r=1}^{n} k^r = \frac{k}{1-k}\left(1-k^n\right)$$

050U $\boxed{k+k^2+k^3+k^4+k^5+k^6+\ldots \quad = \quad \frac{k}{1-k} \ \text{for}\,|k|<1}$

050AU $\boxed{k-k^2+k^3-k^4+k^5-k^6+\ldots = \dfrac{k}{1+k} \text{ for} |k|<1}$

$$S \equiv \sum_{r=1}^{\infty} -(-k)^r \equiv \quad k-k^2+k^3-k^4+k^5-k^6+\ldots$$

$$S = \quad k-k^2+k^3-k^4+k^5-k^6+\ldots$$

$$k \cdot S_n = \qquad k^2-k^3+k^4-k^5+k^6-k^7+\ldots$$

$$S_n + k \cdot S_n = k$$

$$(1+k) \cdot S_n = \quad k$$

$$\therefore \sum_{r=1}^{n} -(-k)^r = \frac{k}{1+k} \text{ for } |k|<1$$

$$S_n = \quad 1+k+k^2+k^3+k^4+\ldots+k^{n-1}$$

$$k \cdot S_n = \qquad k+k^2+k^3+k^4+k^5+\ldots+k^n$$

$$S_n - k \cdot S_n = 1-k^n$$

$$(1-k) \cdot S_n = \quad 1-k^n$$

$$\therefore S_n = \quad \frac{1-k^n}{1-k}$$

051F $\boxed{1+k+k^2+k^3+k^4+\ldots+k^{n-1} = \dfrac{1-k^n}{1-k}}$

051U $\boxed{1+k+k^2+k^3+k^4+k^5+k^6+\ldots = \dfrac{1}{1-k} \text{ for} |k|<1}$

$$1+\frac{k}{1-k} = \frac{(1-k)+k}{1-k} = \frac{1}{1-k}$$

051AF $\boxed{1-k+k^2-k^3+k^4-k^5+\ldots+(-k)^{n-1} = \dfrac{1-(-k)^n}{1+k}}$

$$S_n = \quad 1-k+k^2-k^3+k^4-\ldots+(-k)^{n-1}$$

$$k \cdot S_n = \qquad k-k^2+k^3-k^4+k^5+\ldots-(-k)^n$$

$$S_n + k \cdot S_n = 1-(-k)^n$$

$$(1+k) \cdot S_n = \quad 1-(-k)^n$$

$$\therefore S_n = \quad \frac{1-(-k)^n}{1+k}$$

051AU $\boxed{1-k+k^2-k^3+k^4-k^5+k^6-\ldots = \dfrac{1}{1+k} \text{ for} |k|<1}$

$$1-\frac{k}{1+k} = \frac{(1+k)-k}{1+k} = \frac{1}{1+k}$$

If k = 1 then the series is not convergent. The manipulations we have done above are then not valid. Some researchers have used the idea of evaluating the series in an "uncritical way" to state that:

$$1-1+1-1+1-1+1-1+1-\ldots \overset{c}{=} \frac{1}{1+1} = \tfrac{1}{2}$$

In this case the equals sign should be modified to show that there is something special going on with such an assignment.

Such assignments were made by Euler, but without strong foundations. They were however experimentally found to be useful in his researches.

$$\boxed{1-2+2^2-2^3+2^4-2^5+2^6-\ldots \overset{E}{=} \frac{1}{1+2} = \tfrac{1}{3}}$$

Clearly such an infinite oscillatory divergent series does not have a sum in the ordinary sense. Euler therefore felt it was ok to *define* the infinite sum, since there would be no conflict.[1]

055U $$\boxed{1+2x+3x^2+4x^3+5x^4+6x^5+\ldots = \frac{1}{(1-x)^2} \quad \text{for} \, |x|<1}$$

$$\sum_{r=1}^{\infty} r \cdot x^{r-1} \equiv 1+2x+3x^2+4x^3+5x^4+6x^5+\ldots$$

$$\begin{aligned}
S &= 1+2x+3x^2+4x^3+5x^4+6x^5+\ldots \\
x \cdot S &= \quad x+2x^2+3x^3+4x^4+5x^5+\ldots \\
S - x \cdot S &= 1+x+ \quad x^2+x^3+ \quad x^4+ \quad x^5+\ldots \\
x(1-x)S &= \quad x+ \quad x^2+x^3+ \quad x^4+ \quad x^5+\ldots \\
(1-x)(1-x)S &= 1 \\
\therefore S &= \frac{1}{(1-x)^2}
\end{aligned}$$

P826

$|x|<1$ is a requirement because we have two steps where we discard the last term, if we consider the finite sum. Such discards are required to be vanishingly small, which only happens for high powers of below unity magnitudes.

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[1] *An Introduction to the Theory of Infinite Series*, Chapter XI, T.J.Bromwich, 1907 (2020 facsimile copy, Alpha Editions).

055AU $\boxed{1-2x+3x^2-4x^3+5x^4-6x^5+\ldots = \quad \frac{1}{(1+x)^2} \ \text{for} |x|<1}$

$$\sum_{r=1}^{\infty} r\cdot(-x)^{r-1} \equiv \quad 1-2x+3x^2-4x^3+5x^4-6x^5+\ldots$$

$$S = \quad 1-2x+3x^2-4x^3+5x^4-6x^5+\ldots$$

$$x\cdot S = \qquad x+2x^2+3x^3+4x^4+5x^5+\ldots$$

$$S+x\cdot S = \quad 1-x+\ x^2-\ x^3+\ \ x^4-\ x^5+\ldots$$

$$x(1+x)S = \qquad x+x^2+\ x^3+\ \ x^4+\ x^5+\ldots$$

$$(1+x)(1+x)S = \quad 1$$

$$\therefore S = \frac{1}{(1+x)^2}$$

P826

$|x|<1$ is a requirement because we have two steps where we discard the last term, if we consider the finite sum. Such discards are required to be vanishingly small, which only happens for high powers of below unity magnitudes.

060F $\boxed{1\times 1!+2\times 2!+3\times 3!+4\times 4!+5\times 5!+\ldots+n\times n! = \quad (n+1)!-1}$

$$\sum_{r=1}^{n} r\times r! = 1\times 1!+2\times 2!+3\times 3!+4\times 4!+5\times 5!+\ldots+n\times n!$$

$$=(2-1)\times 1!+(3-1)\times 2!+(4-1)\times 3!+(5-1)\times 4!+\ldots+((n+1)-1)\times n!$$

$$=(-1!+2!)+(-2!+3!)+(-3!+4!)+(-4!+5!)+\ldots+(-n!+(n+1)!)$$

We have created a *telescoping sum* where adjacent terms cancel in pairs, leaving only the first and last terms unmatched.

$$\therefore \sum_{r=1}^{n} r\times r! = (n+1)!-1$$

J288

070F $\boxed{1^2+2^2+3^2+4^2+5^2+6^2+\ldots+n^2 = \quad \frac{n}{6}(2n+1)(n+1)}$

$$\sum_{r=1}^{n} r^2 \equiv \quad 1^2+2^2+3^2+4^2+5^2+6^2+\ldots+n^2$$ sum of squares

$$\sum_{r=1}^{1} r^2 = \quad 1 \qquad \sum_{r=1}^{2} r^2 = \quad 5 \qquad \sum_{r=1}^{3} r^2 = \quad 14$$

Assume that: $$\sum_{r=1}^{n} r^2 = \quad an^3+bn^2+cn$$

This is reasonable because $$\sum_{r=1}^{n} r = \quad 0.5n^2+0.5n$$

For $n=1$ $a+b+c=1$
For $n=2$ $8a+4b+2c=5$
For $n=3$ $27a+9b+3c=14$

Eliminate *c* by multiplication and subtraction of the first equation:

$$6a+2b=3$$
$$24a+6b=11$$

Eliminate *b* by ×3 then subtraction:

$$6a=2$$

Back substitution:

$$2b=1$$

$$6a+6b+6c=6 \qquad 6c=6-2-3=1$$

The suggestion is that: $$\sum_{r=1}^{n} r^2 = \frac{n}{6}(2n+1)(n+1)$$ J019

The formula was shown to work for *n* = 1, 2, and 3.

To prove that it works for all *n* we use *mathematical induction*.

We only require: $$\sum_{r=1}^{n+1} r^2 = (n+1)^2 + \sum_{r=1}^{n} r^2$$

$$\sum_{r=1}^{n+1} r^2 = \frac{(n+1)}{6}(2n+3)(n+2) = \frac{(n+2)}{6}(2n+3)(n+1)$$

$$= \frac{n}{6}(2n+3)(n+1) + \frac{2}{6}(2n+3)(n+1)$$

$$\frac{n}{6}(2n+3)(n+1) = \frac{n}{6}(2n+1)(n+1) + \frac{n}{6}(2)(n+1)$$

$$\sum_{k=1}^{n+1} r^2 = \left(\sum_{k=1}^{n} r^2\right) + \frac{2}{6}(2n+3)(n+1) + \frac{2n}{6}(n+1)$$

$$\frac{2}{6}(2n+3)(n+1) + \frac{2n}{6}(n+1) = \frac{2}{6}\left[(2n+3)(n+1)+n(n+1)\right]$$

$$= \frac{1}{3}\left(2n^2+2n+3n+3+n^2+n\right)$$

$$= \frac{1}{3}\left(3n^2+6n+3\right) = n^2+2n+1 = (n+1)^2$$

which completes the proof by induction.

080F $\boxed{1^3+2^3+3^3+4^3+5^3+6^3+\ldots+n^3 = \quad \frac{n^2}{4}(n+1)^2}$

$$\sum_{r=1}^{n} r^3 \equiv \quad 1^3+2^3+3^3+4^3+5^3+6^3+\ldots+n^3$$ sum of cubes

$$\sum_{r=1}^{1} r^3 = \quad 1 \qquad\qquad \sum_{r=1}^{2} r^3 = \quad 9 \qquad\qquad \sum_{r=1}^{3} r^3 = \quad 36$$

Assume that: $$\sum_{r=1}^{n} r^3 = \quad an^4+bn^3+cn^2$$

This is not *totally* un-reasonable because $\sum_{r=1}^{n} r$ has a second-order sum

and $\sum_{r=1}^{n} r^2$ has a third-order sum.

Also, an integral (a continuous sum) increases the power by 1.

For $n=1$ $\quad a+b+c=1$
For $n=2$ $\quad 16a+8b+4c=9$
For $n=3$ $\quad 81a+27b+9c=36$

Eliminate *c* by multiplication and subtraction of the first equation:

$$12a+4b=5$$
$$72a+18b=27$$

Eliminate *a* by ×6 then subtraction: $6b=3$ $\quad b = \frac{1}{2}$
Back substitution: $12a=3$ $\quad a = \frac{1}{4}$
$12a+12b+12c=12$ $\quad 12c=12-3-6=3$ $\quad c = \frac{1}{4}$

The suggestion is that: $$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}\left(n^2+2n+1\right)$$ J020

The formula was shown to work for *n* = 1, 2, and 3.

To prove that it works for all *n* we use *mathematical induction*.

We only require: $$\sum_{r=1}^{n+1} r^3 = (n+1)^3 + \sum_{r=1}^{n} r^3$$

In other words: $$\left(\sum_{r=1}^{n+1} r^3\right) - \left(\sum_{r=1}^{n} r^3\right) = (n+1)^3$$

$$\sum_{r=1}^{n+1} r^3 = \frac{(n+1)^2}{4}\left[(n+1)^2+2n+3\right] = \frac{n^2+(2n+1)}{4}\left[\left(n^2+2n+1\right)+(2n+3)\right]$$

$$\left(\sum_{r=1}^{n+1} r^3\right) - \left(\sum_{r=1}^{n} r^3\right) = \frac{(2n+1)}{4}\left[\left(n^2+2n+1\right)+(2n+3)\right] + \frac{n^2}{4}(2n+3)$$

$$= \frac{1}{4}\left[(2n+1)\left(n^2+4n+4\right)+\left(2n^3+3n^2\right)\right]$$

$$= \frac{1}{4}\left[\left(2n^3+8n^2+8n\right)+\left(n^2+4n+4\right)+\left(2n^3+3n^2\right)\right]$$

$$= \frac{1}{4}\left[4n^3+12n^2+12n+4\right] \quad = \quad n^3+3n^2+3n+1$$

$$= (n+1)^3$$

which completes the proof by induction.

082F $\boxed{\dfrac{0+1+2+3+\ldots+n}{n+n+n+n+\ldots+n}=\dfrac{1}{2}}$

Using 010F, the numerator sums to $\frac{1}{2}n\left(n+1\right)$.

The denominator has (n+1) lots of n. The result of the division yields ½.

084FE $\boxed{\dfrac{0^2+1^2+2^2+3^2+4^2+\ldots+n^2}{n^2+n^2+n^2+n^2+n^2+\ldots+n^2}\cong\dfrac{1}{3}\ \text{for large } n}$

Using 070F, the numerator sum $\frac{n}{6}(2n+1)(n+1)$

simplifies to $\left(n+1\right)\frac{n^2}{3}$ for large n.

The denominator sum is $\left(n+1\right)n^2$, and the result follows by division.

086FE $\boxed{\dfrac{0^3+1^3+2^3+3^3+4^3+\ldots+n^3}{n^3+n^3+n^3+n^3+n^3+\ldots+n^3}\cong\dfrac{1}{4}\ \text{for large } n}$

Using 080F, the numerator sum $\frac{n^2}{4}(n+1)^2$

simplifies to $\left(n+1\right)\frac{n^3}{4}$ for large n.

The denominator sum is $\left(n+1\right)n^3$, and the result follows by division.

090FE $\boxed{\dfrac{0^m+1^m+2^m+3^m+4^m+\ldots+n^m}{n^m+n^m+n^m+n^m+n^m+\ldots+n^m}\cong\dfrac{1}{m+1}\ \text{for large } n}$ $m\in\mathbb{N}$

The denominator sum is $\left(n+1\right)n^m$

The numerator sum can be found using Faulhaber's formula, although we only require the multiplier of the highest power of n in the polynomial.

In other words we should like to prove that:

$$0^m+1^m+2^m+3^m+4^m+\ldots+n^m\cong\frac{n^{m+1}}{m+1}\quad\text{for large } n\qquad m\in\mathbb{N}$$

Sadly we don't haven't found a simple proof for this.

The result of 090FE is attributed to Wallis (*Arithmetica Infinitorum* - 1656) in *Infinitesimal* (Chapter 9) by Amir Alexander (2014), although the method used was effectively to see the progression of 082F, 084FE, and 086FE, {**1/2** for m=1; **1/3** for m=2; **1/4** for m=3} and make an educated guess.

100F $$\boxed{\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots+\frac{1}{k^n}= \quad \frac{1}{(k-1)}\left(1-\frac{1}{k^n}\right)}$$

$$S_n \equiv \sum_{r=1}^{n}\frac{1}{k^r} \equiv \quad \frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots+\frac{1}{k^n}$$

$$S_n = \quad \frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots+\frac{1}{k^{n-1}}+\frac{1}{k^n}$$

$$k\cdot S_n = \quad 1+\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots+\frac{1}{k^{n-1}}$$

$$k\cdot S_n - S_n = \quad 1-\frac{1}{k^n}$$

$$(k-1)\cdot S_n = \quad 1-\frac{1}{k^n}$$

$$\therefore \sum_{r=1}^{n}\frac{1}{k^r} = \quad \frac{1}{(k-1)}\left(1-\frac{1}{k^n}\right)$$

100U $$\boxed{\frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\frac{1}{k^5}+\frac{1}{k^6}+\ldots= \quad \frac{1}{(k-1)} \qquad \text{for } |k|>1}$$

100UE $$\boxed{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\ldots= \quad 1}$$

100AU $$\boxed{\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\frac{1}{k^6}+\ldots= \quad \frac{1}{(k+1)} \qquad \text{for } |k|>1}$$

$$S_n \equiv \sum_{r=1}^{n}\frac{(-1)^{r+1}}{k^r} \equiv \quad \frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\frac{1}{k^6}+\ldots-\frac{(-1)^n}{k^n}$$

$$S_n = \quad \frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\frac{1}{k^6}+\ldots-\frac{(-1)^n}{k^n}$$

$$\frac{S_n}{k} = \quad +\frac{1}{k^2}-\frac{1}{k^3}+\frac{1}{k^4}-\frac{1}{k^5}+\frac{1}{k^6}+\ldots+\frac{(-1)^n}{k^n}-\frac{(-1)^n}{k^{n+1}}$$

$$S_n+\frac{S_n}{k} = \quad \frac{1}{k}-\frac{(-1)^n}{k^{n+1}}$$

$$\frac{(k+1)S_n}{k} = \quad \frac{1}{k}-\frac{(-1)^n}{k^{n+1}} \qquad\qquad \therefore S_n = \quad \frac{1}{k+1}\left(1-\frac{(-1)^n}{k^n}\right)$$

100AF $$\boxed{\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\ldots-\frac{(-1)^n}{k^n}= \quad \frac{1}{k+1}\left(1-\frac{(-1)^n}{k^n}\right)}$$

100AFE $$\boxed{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\ldots-\frac{(-1)^n}{2^n}= \quad \frac{1}{3}\left(1-\frac{(-1)^n}{2^n}\right)}$$

100AUE $$\boxed{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\frac{1}{2^6}+\ldots= \quad \frac{1}{3}}$$

110AU $\boxed{\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots= \quad \ln(2)}$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots$$

We can only evaluate this one ‘backwards’:

For $\quad f(x)=\ln(1+x)$ $\quad$ *ln*( ) being the natural logarithm function

$$\frac{d}{dx}\cdot f(x)=f'(x)=\frac{1}{(1+x)}=+(1+x)^{-1}$$

$$\frac{d}{dx}\cdot f'(x)=f''(x)=-(1+x)^{-2}$$

$$\frac{d}{dx}\cdot f''(x)=f'''(x)=+2(1+x)^{-3}$$

$$\frac{d}{dx}\cdot f'''(x)=f''''(x)=-3\cdot 2(1+x)^{-4}$$

Maclaurin’s expansion is:

$$f(x)=f(0)+x\cdot f'(0)+\frac{x^2}{2!}\cdot f''(0)+\frac{x^3}{3!}\cdot f'''(0)+\frac{x^4}{4!}\cdot f''''(0)+\ldots$$

valid for $-1<x\le+1$

$$\ln(1+x)=0+x-\frac{x^2}{2!}+\frac{x^3}{3!}\cdot 2-\frac{x^4}{4!}\cdot 3\cdot 2+\ldots$$

$$\ln(1+1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln(2)$$

120UE $\boxed{\frac{2}{3}+\frac{3}{3^2}+\frac{2}{3^3}+\frac{3}{3^4}+\frac{2}{3^5}+\frac{3}{3^6}+\frac{2}{3^7}+\ldots= \quad \frac{9}{8}}$

$$\sum_{r=1}^{\infty}\frac{1}{2\times 3^r}\left(5+(-1)^r\right)=\frac{9}{8}$$

This one just serves as an example of how to solve such a series. It was found in Jolley’s collection (J013), but is referenced back to a book from 1899.

$$S=\frac{2}{3}+\frac{3}{3^2}+\frac{2}{3^3}+\frac{3}{3^4}+\frac{2}{3^5}+\frac{3}{3^6}+\frac{2}{3^7}+\ldots$$

$$=2\times\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\frac{1}{3^6}+\ldots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\frac{1}{3^8}+\ldots\right)$$

$$=2\times\left(\frac{1}{3-1}\right)+\left(\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\frac{1}{9^4}+\frac{1}{9^5}+\ldots\right)$$

$$=1+\frac{1}{9-1}= \quad \frac{9}{8}$$

121AUE $\boxed{\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\frac{4}{7^5}-\frac{5}{7^6}+\frac{4}{7^7}-\ldots=\quad\frac{23}{48}}$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{2\times 7^r}\left(9+(-1)^r\right)=\frac{23}{48}$$

This one just serves as an example of how to solve such a series. It was found in Jolley's collection (J014), but is referenced back to a book from 1899.

$$S=\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\frac{4}{7^5}-\frac{5}{7^6}+\frac{4}{7^7}-\ldots$$

$$=4\times\left(\frac{1}{7}-\frac{1}{7^2}+\frac{1}{7^3}-\frac{1}{7^4}+\frac{1}{7^5}-\ldots\right)-\left(\frac{1}{7^2}+\frac{1}{7^4}+\frac{1}{7^6}+\frac{1}{7^8}+\ldots\right)$$

$$=4\times\left(\frac{1}{7+1}\right)-\left(\frac{1}{49}+\frac{1}{49^2}+\frac{1}{49^3}+\frac{1}{49^4}+\ldots\right)$$

$$=\frac{1}{2}-\left(\frac{1}{49-1}\right)=\quad\frac{23}{48}$$

140AUE $\boxed{1-\frac{1}{3^2}+\frac{1}{3^4}-\frac{1}{3^6}+\frac{1}{3^8}-\frac{1}{3^{10}}+\ldots=\quad\frac{9}{10}}$

$$\sum_{r=0}^{\infty}\frac{(-1)^r}{3^{2r}}=\quad\frac{9}{10}$$

$$S=\quad 1-\frac{1}{3^2}+\frac{1}{3^4}-\frac{1}{3^6}+\frac{1}{3^8}-\frac{1}{3^{10}}+\ldots$$

$$\frac{S}{3^2}=\quad\frac{1}{3^2}-\frac{1}{3^4}+\frac{1}{3^6}-\frac{1}{3^8}+\frac{1}{3^{10}}-\ldots$$

$$\left(1+\frac{1}{3^2}\right)S=1\qquad\qquad\therefore S=\frac{3^2}{1+3^2}=\frac{9}{10}$$

$$S=\quad 1-\frac{1}{k^2}+\frac{1}{k^4}-\frac{1}{k^6}+\frac{1}{k^8}-\frac{1}{k^{10}}+\ldots$$

$$\frac{S}{k^2}=\quad\frac{1}{k^2}-\frac{1}{k^4}+\frac{1}{k^6}-\frac{1}{k^8}+\frac{1}{k^{10}}-\ldots$$

$$\left(1+\frac{1}{k^2}\right)S=1\qquad\qquad\therefore S=\frac{k^2}{1+k^2}$$

140AU $\boxed{1-\frac{1}{k^2}+\frac{1}{k^4}-\frac{1}{k^6}+\frac{1}{k^8}-\frac{1}{k^{10}}+\ldots=\quad\frac{k^2}{1+k^2}\ \text{ for}\,|k|>1}$

150FE $\boxed{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}=\quad 1-\frac{1}{n+1}}$

$$\sum_{r=1}^{n}\frac{1}{r(r+1)}\equiv\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}$$

First use a *partial fraction* expansion, $a_r=\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$

Then compare terms:

$$a_1=\frac{1}{1}-\frac{1}{2}\qquad a_2=\frac{1}{2}-\frac{1}{3}\qquad a_3=\frac{1}{3}-\frac{1}{4}\qquad a_4=\frac{1}{4}-\frac{1}{5}\qquad \ldots$$

$$a_{n-1}=\frac{1}{n-1}-\frac{1}{n}\qquad\qquad a_n=\frac{1}{n}-\frac{1}{n+1}$$

When written this way, it is evident that there are pairs of equal size and opposite sign that cancel when the series is summed, leaving just the first and last terms remaining. This is known as a *telescoping sum*, much like a sliding tube telescope that can be closed up.

It should be noted that individually each of the parts which have been created (using the partial fractions method) have infinite sums. However, given that the finite sum has a definite value, and the left-over part becomes vanishingly small as *n* tends to infinity, the infinite sum has a simple limit.

150UE $\boxed{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\ldots=\quad 1}$

155FE $\boxed{\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)}$

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}\equiv\quad \frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{n(n+2)}$$

Using partial fractions: $\frac{2}{r(r+2)}=\quad\frac{1}{r}-\frac{1}{r+2}$

$$\sum_{r=1}^{n}\frac{1}{r}=\quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n}$$

$$\sum_{r=1}^{n}\frac{1}{r+2}=\qquad \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}$$

$$2\times\sum_{r=1}^{n}\frac{1}{r(r+2)}=\left(\sum_{r=1}^{n}\frac{1}{r}\right)-\left(\sum_{r=1}^{n}\frac{1}{r+2}\right)$$

$$=\quad\frac{1}{1}+\frac{1}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$$

$$\therefore\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)$$

155U $\boxed{\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots=\quad\frac{3}{4}}$

160U $\boxed{\dfrac{1}{1\cdot(1+k)}+\dfrac{1}{2\cdot(2+k)}+\dfrac{1}{3\cdot(3+k)}+\ldots = \quad \dfrac{1}{k}\cdot\sum_{r=1}^{k}\dfrac{1}{r}}$ ($k \in \mathbb{N}$)

$$\sum_{r=1}^{n}\frac{1}{r\,(r+k)} \equiv \quad \frac{1}{1\cdot(1+k)}+\frac{1}{2\cdot(2+k)}+\frac{1}{3\cdot(3+k)}+\frac{1}{4\cdot(4+k)}+\ldots$$

Using partial fractions: $\dfrac{k}{r\,(r+k)} = \quad \dfrac{1}{r}-\dfrac{1}{r+k}$

$$\sum_{r=1}^{\infty}\frac{1}{r} = \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{k}+\frac{1}{(1+k)}+\ldots$$

$$\sum_{r=1}^{\infty}\frac{1}{r+k} = \qquad\qquad \frac{1}{(1+k)}+\frac{1}{(2+k)}+\frac{1}{(3+k)}+\ldots$$

$$k\times\sum_{r=1}^{\infty}\frac{1}{r\,(r+k)} = \left(\sum_{r=1}^{\infty}\frac{1}{r}\right)-\left(\sum_{r=1}^{\infty}\frac{1}{r+k}\right)$$

$$= \sum_{r=1}^{k}\frac{1}{r}$$

notice the change of the upper limit

$$\therefore \sum_{r=1}^{\infty}\frac{1}{r\,(r+k)} = \frac{1}{k}\cdot\sum_{r=1}^{k}\frac{1}{r}$$

160AU $\boxed{\dfrac{1}{1\cdot 2}-\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}-\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}-\ldots = \quad 2\cdot\ln(2)-1}$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r\cdot(r+1)} \equiv \frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}-\ldots$$

Using partial fractions: $\dfrac{1}{r\,(r+1)} = \dfrac{1}{r}-\dfrac{1}{r+1}$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r\cdot(r+1)} = \left(\frac{1}{1}-\frac{1}{2}\right)-\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)-\ldots$$

$$= 1+2\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\ldots\right)$$

$$= 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\ldots\right)+1-2$$

$$= 2\cdot\ln(2)-1 \quad = \quad \ln(4)-1$$

170AU $\boxed{\dfrac{3}{1\cdot 2}-\dfrac{5}{2\cdot 3}+\dfrac{7}{3\cdot 4}-\dfrac{9}{4\cdot 5}+\dfrac{11}{5\cdot 6}-\ldots \quad = \quad 1}$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}(2r+1)}{r\cdot(r+1)} \equiv \frac{3}{1\cdot 2}-\frac{5}{2\cdot 3}+\frac{7}{3\cdot 4}-\frac{9}{4\cdot 5}+\frac{11}{5\cdot 6}-\ldots$$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}(2r+1)}{r\cdot(r+1)} = \left(2\times\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{(r+1)}\right)+\left(\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r\cdot(r+1)}\right)$$

$$= 2\cdot\left(\sum_{r=2}^{\infty}\frac{(-1)^{r}}{r}\right)+2\cdot\ln(2)-1$$

$$= -2\cdot\left(\sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{r}\right)+2\cdot\ln(2)-1$$

$$= -2\cdot\left(-1+\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\right)+2\cdot\ln(2)-1$$

$$= -2\cdot(\ln(2)-1)+2\cdot\ln(2)-1$$

$$= 2-1 \quad = \quad 1$$

200F $\boxed{\dfrac{1}{2^2-1}+\dfrac{1}{3^2-1}+\dfrac{1}{4^2-1}+\ldots+\dfrac{1}{n^2-1} = \quad \dfrac{3}{4}-\dfrac{1}{2}\left(\dfrac{1}{n}+\dfrac{1}{n+1}\right)}$

$$\sum_{r=2}^{n}\frac{1}{r^2-1} \equiv \quad \frac{1}{2^2-1}+\frac{1}{3^2-1}+\frac{1}{4^2-1}+\frac{1}{5^2-1}+\ldots+\frac{1}{n^2-1}$$

Using partial fractions: $\dfrac{2}{r^2-1} = \quad \dfrac{1}{r-1}-\dfrac{1}{r+1}$

$$\sum_{r=2}^{n}\frac{1}{r-1} = \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n-1}$$

$$\sum_{r=2}^{n}\frac{1}{r+1} = \qquad\qquad \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}$$

$$2\times\sum_{r=2}^{n}\frac{1}{r^2-1} = \left(\sum_{r=2}^{n}\frac{1}{r-1}\right)-\left(\sum_{r=2}^{n}\frac{1}{r+1}\right)$$

$$= \quad \frac{1}{1}+\frac{1}{2} \quad - \quad \left(\frac{1}{n}+\frac{1}{n+1}\right)$$

$$\therefore \sum_{r=2}^{n}\frac{1}{r^2-1} = \frac{3}{4}-\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)$$

200U $\boxed{\dfrac{1}{2^2-1}+\dfrac{1}{3^2-1}+\dfrac{1}{4^2-1}+\dfrac{1}{5^2-1}+\dfrac{1}{6^2-1}+\ldots = \quad \dfrac{3}{4}}$

300U $\boxed{\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots= \quad \exp(1) \quad =e}$

Consider $\quad g(x)=\frac{x^n}{n!} \qquad \frac{d(\,g(x)\,)}{dx}=\frac{n\cdot x^{n-1}}{n\,!}=\frac{x^{n-1}}{(n-1)!}$

The term is shifted down to a lower power, and a lower factorial.

Then with $\quad f(x)=\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots+\frac{x^n}{n!}+\ldots$

$$\frac{d(f(x))}{dx}=\frac{1\cdot x^{1-1}}{1!}+\frac{2\cdot x^{2-1}}{2!}+\frac{3\cdot x^{3-1}}{3!}+\frac{4\cdot x^{4-1}}{4!}+\frac{5\cdot x^{5-1}}{5!}+\ldots+\frac{n\cdot x^{n-1}}{n!}+\ldots$$

$$=\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots+\frac{x^n}{n!}+\ldots$$

This works because $\quad 0!\equiv 1!$

$$\frac{d(f(x))}{dx}=f(x)$$

The derivative is equal to the function.

We also require $f(0)=1$. With both conditions satisfied we have *created* the exponential function. There is another little tricky part here. We also require: $\quad 0^0=1$. $\quad$ In many respects it is easier to say:

J097, J119

$$1+\sum_{r=1}^{\infty}\frac{1}{r!}=\exp(1)=e$$

avoiding 0! and 0^0

$$2+\sum_{r=2}^{\infty}\frac{1}{r!}=\exp(1)=e$$

avoiding 1!

310F $\boxed{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots+\frac{n-1}{n!}= \quad 1-\frac{1}{n\,!}}$

$$\sum_{r=2}^{n}\frac{r-1}{r!}\equiv\frac{2-1}{2!}+\frac{3-1}{3!}+\frac{4-1}{4!}+\ldots+\frac{n-1}{n!}$$

$$=\left(\frac{2}{2!}-\frac{1}{2!}\right)+\left(\frac{3}{3!}-\frac{1}{3!}\right)+\left(\frac{4}{4!}-\frac{1}{4!}\right)+\ldots+\left(\frac{n}{n!}-\frac{1}{n!}\right)$$

$$=\left(\frac{1}{(2-1)!}-\frac{1}{2!}\right)+\left(\frac{1}{(3-1)!}-\frac{1}{3!}\right)+\left(\frac{1}{(4-1)!}-\frac{1}{4!}\right)+\ldots+\left(\frac{1}{(n-1)!}-\frac{1}{n!}\right)$$

$$=\frac{1}{1!}-\frac{1}{2!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}-\frac{1}{4!}+\ldots+\frac{1}{(n-1)!}-\frac{1}{n!}$$

$$=1-\frac{1}{n!}$$

due to the *telescoping sum*

$$\therefore\sum_{r=2}^{n}\frac{r-1}{r!}=1-\frac{1}{n\,!}$$

310U $\boxed{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots= \quad 1}$

J302

These power series are created by repeated differentiations of the given functions. Only the few which have simple series have been given below:

320U $$\boxed{x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\ldots=\quad\sinh(x)}$$

$$\sinh(x)=\tfrac{1}{2}\left(e^x-e^{-x}\right)$$ P830 / P864

320AU $$\boxed{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots=\quad\sin(x)}$$

P846

330U $$\boxed{x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}-\ldots=\quad\tanh^{-1}(x)\qquad\text{for}\,|x|<1}$$

P871

330AU $$\boxed{x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\ldots=\quad\tan^{-1}(x)\qquad\text{for}\,|x|<1}$$

P853

340U $$\boxed{1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}+\ldots=\quad\cosh(x)}$$

$$\cosh(x)=\tfrac{1}{2}\left(e^x+e^{-x}\right)$$ P829 / P865

340AU $$\boxed{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots=\quad\cos(x)}$$

P847

350AU $$\boxed{1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}-\frac{x^{10}}{5!}+\ldots=\quad\exp\left(-x^2\right)}$$

P831

360U $$\boxed{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots=\quad\exp(x)}$$

P827

370AU $$\boxed{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\ldots=\ln(1+x)\quad\text{for}\,|x|<1}$$

P842

380U $$\boxed{1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\frac{6^3}{6!}+\ldots=\quad 5e}$$

the calculations are shown below 384U J161

384U $$\boxed{\frac{3^2}{2!}+\frac{4^2}{3!}+\frac{5^2}{4!}+\frac{6^2}{5!}+\frac{7^2}{6!}+\ldots=\quad 5(e-1)}$$

$$S\equiv 1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\frac{6^3}{6!}+\ldots\qquad[380\text{U}]$$

$$=1+\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\frac{5^2}{4!}+\frac{6^2}{5!}+\ldots\qquad[384\text{U}]+5$$

$$=1+\frac{(1+1)2}{1!}+\frac{(2+1)3}{2!}+\frac{(3+1)4}{3!}+\frac{(4+1)5}{4!}+\ldots$$

It is convenient to split the series into two parts at this point:

$$S=A+B$$

$$A=1+\frac{1\cdot 2}{1!}+\frac{2\cdot 3}{2!}+\frac{3\cdot 4}{3!}+\frac{4\cdot 5}{4!}+\frac{5\cdot 6}{5!}+\frac{6\cdot 7}{6!}+\ldots$$

$$=1+2+\frac{3}{1!}+\frac{4}{2!}+\frac{5}{3!}+\frac{6}{4!}+\frac{7}{5!}+\frac{8}{6!}+\ldots$$

$$=6+\frac{4}{2!}+\frac{5}{3!}+\frac{6}{4!}+\frac{7}{5!}+\frac{8}{6!}+\ldots$$

$$=6+\frac{3+1}{2!}+\frac{3+2}{3!}+\frac{3+3}{4!}+\frac{3+4}{5!}+\frac{3+5}{6!}+\ldots$$

$$=6+3\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\right)+\left(\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots\right)$$

$$=6+3(e-2)+1 \quad =3e+1$$

Note use of **[300U]** and **[310U]** above and below

$$B=\frac{2}{1!}+\frac{3}{2!}+\frac{4}{3!}+\frac{5}{4!}+\frac{6}{5!}+\frac{7}{6!}+\ldots$$

$$=2+\frac{3}{2!}+\frac{4}{3!}+\frac{5}{4!}+\frac{6}{5!}+\frac{7}{6!}+\ldots$$

$$=2+\frac{2+1}{2!}+\frac{2+2}{3!}+\frac{2+3}{4!}+\frac{2+4}{5!}+\frac{2+5}{6!}+\ldots$$

$$=2+2\left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\ldots\right)+\left(\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots\right)$$

$$=2+2(e-2)+1= \quad 2e-1$$

$$\therefore S=A+B= \quad (3e+1)+(2e-1)= \quad 5e$$

400U $$\boxed{\frac{1\cdot 4^2}{3!}+\frac{2\cdot 5^2}{4!}+\frac{3\cdot 6^2}{5!}+\frac{4\cdot 7^2}{6!}+\frac{5\cdot 8^2}{7!}+\ldots= \quad 6}$$

$$\sum_{r=1}^{\infty}\frac{r(r+3)^2}{(r+2)!}=\frac{1\cdot 4^2}{3!}+\frac{2\cdot 5^2}{4!}+\frac{3\cdot 6^2}{5!}+\frac{4\cdot 7^2}{6!}+\ldots=\sum_{r=3}^{\infty}\frac{(r-2)(r+1)^2}{r!}$$

$$\sum_{r=3}^{\infty}\frac{(r-2)(r+1)^2}{r!}=\left(\sum_{r=3}^{\infty}\frac{r^3}{r!}\right)-3\cdot\left(\sum_{r=3}^{\infty}\frac{r}{r!}\right)-2\cdot\left(\sum_{r=3}^{\infty}\frac{1}{r!}\right)$$

$$\sum_{r=3}^{\infty}\frac{r^3}{r!}=5(e-1)$$

$$\sum_{r=3}^{\infty}\frac{r}{r!}=\sum_{r=3}^{\infty}\frac{1}{(r-1)!}=\sum_{r=2}^{\infty}\frac{1}{r!}=e-2$$

$$\sum_{r=3}^{\infty}\frac{1}{r!}=e-\frac{5}{2}$$

$$\therefore\sum_{r=1}^{\infty}\frac{r(r+3)^2}{(r+2)!}=(5e-5)-(3e-6)-(2e-5)=6$$

used in the solution of 410U

410U

$$\boxed{1+2^3+\frac{3^3}{2!}+\frac{4^3}{3!}+\frac{5^3}{4!}+\frac{6^3}{5!}+\frac{7^3}{6!}+\ldots= \quad 15e}$$

J101

$$\sum_{r=0}^{\infty}\frac{(r+1)^3}{r!}\equiv \quad \frac{1^3}{0!}+\frac{2^3}{1!}+\frac{3^3}{2!}+\frac{4^3}{3!}+\frac{5^3}{4!}+\frac{6^3}{5!}+\frac{7^3}{6!}+\ldots$$

$$\begin{aligned}
S &= 1+2^3+\frac{3^3}{2!}+\frac{4^3}{3!}+\frac{5^3}{4!}+\frac{6^3}{5!}+\frac{7^3}{6!}+\ldots \\
&= 9+\frac{(0+3)3^2}{2!}+\frac{(1+3)4^2}{3!}+\frac{(2+3)5^2}{4!}+\frac{(3+3)6^2}{5!}+\frac{(4+3)7^2}{6!}+\ldots \\
&= 9+\left(\frac{1\cdot 4^2}{3!}+\frac{2\cdot 5^2}{4!}+\frac{3\cdot 6^2}{5!}+\frac{4\cdot 7^2}{6!}+\ldots\right)+3\cdot\left(\frac{3^2}{2!}+\frac{4^2}{3!}+\frac{5^2}{4!}+\frac{6^2}{5!}+\frac{7^2}{6!}+\ldots\right) \\
&= 9+(6)+(15e-15) \\
&= 15e
\end{aligned}$$

makes use of **[400U]** *and* **[384U]**

420F

$$\boxed{\frac{1}{k^1}+\frac{2}{k^2}+\frac{3}{k^3}+\ldots+\frac{n}{k^n}= \quad \left(\frac{1}{k-1}\right)\left(1-\frac{n}{k^n}\right)+\left(\frac{1}{k-1}\right)^2\left(1-\frac{1}{k^{n-1}}\right)}$$

Arithmetico-Geometric series

$$S_n\equiv\sum_{r=1}^{n}\frac{r}{k^r}\equiv\frac{1}{k^1}+\frac{2}{k^2}+\frac{3}{k^3}+\frac{4}{k^4}+\ldots+\frac{n}{k^n}$$

$$S_n = \quad \frac{1}{k^1}+\frac{2}{k^2}+\frac{3}{k^3}+\frac{4}{k^4}+\frac{5}{k^5}+\ldots \quad +\frac{n-1}{k^{n-1}}+\frac{n}{k^n}$$

$$k\cdot S_n=\frac{1}{k^0}+\frac{2}{k^1}+\frac{3}{k^2}+\frac{4}{k^3}+\frac{5}{k^4}+\frac{6}{k^5}+\ldots \quad +\frac{n}{k^{n-1}}$$

$$k\cdot S_n-S_n=\frac{1}{k^0}+\frac{2-1}{k^1}+\frac{3-2}{k^2}+\frac{4-3}{k^3}+\frac{5-4}{k^4}+\frac{6-5}{k^5}+\ldots+\frac{n-(n-1)}{k^{n-1}}-\frac{n}{k^n}$$

$$\begin{aligned}
(k-1)\cdot S_n &= 1+\frac{1}{k^1}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\frac{1}{k^5}+\ldots+\frac{1}{k^{n-1}}-\frac{n}{k^n} \\
&= \left(1-\frac{n}{k^n}\right)+\sum_{r=1}^{n-1}\frac{1}{k^r} \\
&= \left(1-\frac{n}{k^n}\right)+\frac{1}{k}\left(\frac{1}{1-\frac{1}{k}}\right)\left(1-\frac{1}{k^{n-1}}\right)
\end{aligned}$$

$$(k-1)\cdot S_n=\left(1-\frac{n}{k^n}\right)+\left(\frac{1}{k-1}\right)\left(1-\frac{1}{k^{n-1}}\right)$$

430FE $\boxed{\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\frac{5}{2^5}+\ldots+\frac{n}{2^n}= \quad 2-\frac{1}{2^{n-1}}-\frac{n}{2^n}}$

430UE $\boxed{\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\frac{5}{2^5}+\ldots= \quad 2}$

$$\therefore S_n=\left(\frac{1}{k-1}\right)\left(1-\frac{n}{k^n}\right)+\left(\frac{1}{k-1}\right)^2\left(1-\frac{1}{k^{n-1}}\right)$$

For $|k|>1$ the series converges:

$$\sum_{r=1}^{\infty}\frac{r}{k^r} \quad =\left(\frac{1}{k-1}\right)+\left(\frac{1}{k-1}\right)^2 \qquad \text{for } k>1$$

$$=\frac{(k-1)+1}{(k-1)^2} \quad = \quad \frac{k}{(k-1)^2}$$

430U $\boxed{\frac{1}{k^1}+\frac{2}{k^2}+\frac{3}{k^3}+\frac{4}{k^4}+\frac{5}{k^5}+\ldots \quad = \quad \frac{k}{(k-1)^2} \qquad \text{for } |k|>1}$

430AU $\boxed{\frac{1}{k}-\frac{2}{k^2}+\frac{3}{k^3}-\frac{4}{k^4}+\frac{5}{k^5}-\ldots= \quad \frac{k}{(k+1)^2}}$

$$S\equiv\sum_{r=1}^{\infty}\frac{r\cdot(-1)^{r+1}}{k^r} \quad = \quad \frac{1}{k}-\frac{2}{k^2}+\frac{3}{k^3}-\frac{4}{k^4}+\frac{5}{k^5}-\ldots$$

$$S= \quad \frac{1}{k}-\frac{2}{k^2}+\frac{3}{k^3}-\frac{4}{k^4}+\frac{5}{k^5}-\ldots$$

$$k\cdot S= \quad 1-\frac{2}{k}+\frac{3}{k^2}-\frac{4}{k^3}+\frac{5}{k^4}-\frac{6}{k^5}+\ldots$$

$$(k+1)S= \quad 1-\frac{1}{k}+\frac{1}{k^2}-\frac{1}{k^3}+\frac{1}{k^4}-\frac{1}{k^5}+\ldots$$

$$\left(\frac{k+1}{k}\right)S= \quad +\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\ldots$$

$$\left(\left(\frac{k+1}{k}\right)+(k+1)\right)S=1$$

$$\left(\frac{1+k^2+2k}{k}\right)S=1$$

$$\therefore\sum_{r=1}^{\infty}\frac{r\cdot(-1)^{r+1}}{k^r}=\frac{k}{k^2+2k+1}=\frac{k}{(k+1)^2}$$

440UE $\boxed{1+\frac{3}{2^1}+\frac{5}{2^2}+\frac{7}{2^3}+\frac{9}{2^4}+\frac{11}{2^5}+\ldots= \quad 6}$

$$S\equiv\sum_{r=0}^{\infty}\frac{2r-1}{2^r}= \quad \frac{1}{2^0}+\frac{3}{2^1}+\frac{5}{2^2}+\frac{7}{2^3}+\frac{9}{2^4}+\frac{11}{2^5}+\ldots$$

$$S= \quad \frac{1}{2^0}+\frac{3}{2^1}+\frac{5}{2^2}+\frac{7}{2^3}+\frac{9}{2^4}+\frac{11}{2^5}+\ldots$$

$$\frac{S}{2}= \qquad \frac{1}{2^1}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\frac{9}{2^5}+\ldots$$

$$S-\frac{S}{2}=1+2\cdot\left(\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\ldots\right)$$

$$\frac{S}{2}=1+2\cdot\left(1\right)$$

$$\therefore S=2\times3= \quad 6$$

J010

450F $\boxed{1+\frac{2}{k}+\frac{3}{k^2}+\frac{4}{k^3}+\ldots+\frac{n}{k^{n-1}}= \quad \left(\frac{k}{k-1}\right)^2\left(1-\frac{n+1}{k^n}+\frac{n}{k^{n+1}}\right)}$

$$S_n\equiv\sum_{r=1}^{n}\frac{r}{k^{r-1}}\equiv \quad 1+\frac{2}{k}+\frac{3}{k^2}+\frac{4}{k^3}+\ldots+\frac{n}{k^{n-1}}$$

$$S_n= \quad 1 \;+\; \frac{2}{k} \;+\; \frac{3}{k^2} \;+\; \frac{4}{k^3} \;+\; \frac{5}{k^4}+\ldots \quad +\frac{n}{k^{n-1}}$$

$$\frac{S_n}{k}= \qquad \frac{1}{k}+ \;\; \frac{2}{k^2} \;+\; \frac{3}{k^3} \;+\; \frac{4}{k^4} \;+\; \frac{5}{k^5}+\ldots \quad +\frac{n}{k^n}$$

$$S_n-\frac{S_n}{k}=1+\frac{2-1}{k}+\frac{3-2}{k^2}+\frac{4-3}{k^3}+\frac{5-4}{k^4}+\ldots+\frac{n-\left(n-1\right)}{k^{n-1}}-\frac{n}{k^n}$$

$$\left(1-\frac{1}{k}\right)\cdot S_n=1+\frac{1}{k}+ \;\; \frac{1}{k^2}+ \;\; \frac{1}{k^3} \;\; +\frac{1}{k^4} \;\; +\frac{1}{k^5}+\ldots+\frac{1}{k^{n-1}}-\frac{n}{k^n}$$

$$\frac{1}{k}\left(1-\frac{1}{k}\right)\cdot S_n= \quad \frac{1}{k}+ \;\; \frac{1}{k^2}+ \;\; \frac{1}{k^3} \;\; +\frac{1}{k^4} \;\; +\frac{1}{k^5}+\ldots+\frac{1}{k^{n-1}}+\frac{1}{k^n}-\frac{n}{k^{n+1}}$$

$$\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k}\right)\cdot S_n=1-\frac{n+1}{k^n}+\frac{n}{k^{n+1}}$$

$$\therefore S_n=\left(\frac{k}{k-1}\right)^2\left(1-\frac{n+1}{k^n}+\frac{n}{k^{n+1}}\right)$$

450U $\boxed{1+\frac{2}{k}+\frac{3}{k^2}+\frac{4}{k^3}+\frac{5}{k^4}+\ldots= \quad \left(\frac{k}{k-1}\right)^2 \quad \text{for}\left|k\right|>1}$

500U $\boxed{\frac{1\cdot 2}{k}+\frac{2\cdot 3}{k^2}+\frac{3\cdot 4}{k^3}+\frac{4\cdot 5}{k^4}+\frac{5\cdot 6}{k^5}+\ldots=\frac{2k^2}{(k-1)^3}}$

$$\sum_{r=1}^{\infty}\frac{r(r+1)}{k^r}=S=\frac{1\cdot 2}{k}+\frac{2\cdot 3}{k^2}+\frac{3\cdot 4}{k^3}+\frac{4\cdot 5}{k^4}+\frac{5\cdot 6}{k^5}+\ldots$$

$$k\cdot S=2+\frac{2\cdot 3}{k}+\frac{3\cdot 4}{k^2}+\frac{4\cdot 5}{k^3}+\frac{5\cdot 6}{k^4}+\ldots$$

$$k\cdot S-S=2+\frac{2\cdot(3-1)}{k}+\frac{3\cdot(4-2)}{k^2}+\frac{4\cdot(5-3)}{k^3}+\frac{5\cdot(6-4)}{k^4}\ldots$$

$$(k-1)S=2\times\left(1+\frac{2}{k}+\frac{3}{k^2}+\frac{4}{k^3}+\frac{5}{k^4}+\frac{6}{k^5}+\ldots\right)$$

$$S=\frac{2}{(k-1)}\times\left(1+\frac{1+1}{k}+\frac{1+2}{k^2}+\frac{1+3}{k^3}+\frac{1+4}{k^4}+\frac{1+5}{k^5}+\ldots\right)$$

$$S=\frac{2}{(k-1)}\times\left(1+\left(\sum_{r=1}^{\infty}\frac{1}{k^r}\right)+\left(\sum_{r=1}^{\infty}\frac{r}{k^r}\right)\right)$$

$$S=\frac{2}{(k-1)}\times\left(1+\left(\frac{1}{k-1}\right)+\left(\frac{k}{(k-1)^2}\right)\right)$$

$$S=2\times\left(\frac{(k-1)^2+(k-1)+k}{(k-1)^3}\right)$$

$$\therefore S=\frac{2k^2}{(k-1)^3}$$

510U $\boxed{1+\frac{1+d}{k}+\frac{1+2d}{k^2}+\frac{1+3d}{k^3}+\ldots=\frac{k}{k-1}+d\cdot\frac{k}{(k-1)^2}}$

$$S\equiv\sum_{r=0}^{\infty}\frac{1+rd}{k^r}=1+\left(\sum_{r=1}^{\infty}\frac{1}{k^r}\right)+d\times\sum_{r=1}^{\infty}\frac{r}{k^r}$$

$$\sum_{r=0}^{\infty}\frac{1+rd}{k^r}=1+\frac{1}{k-1}+d\cdot\frac{k}{(k-1)^2}$$

$$\sum_{r=0}^{\infty}\frac{1+3r}{5^r}=S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\ldots$$

$$\sum_{r=0}^{\infty}\frac{1+3r}{5^r}=\frac{5}{5-1}+3\cdot\frac{5}{(5-1)^2}=\frac{5}{4}+\frac{15}{16}=\frac{35}{16}$$

510UE $\boxed{1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\frac{13}{5^4}+\ldots=\frac{35}{16}}$

520U

$$\boxed{\frac{1^2}{k}+\frac{2^2}{k^2}+\frac{3^2}{k^3}+\frac{4^2}{k^4}+\ldots=\frac{1}{(k-1)}+\frac{3}{(k-1)^2}+\frac{2}{(k-1)^3}\quad \text{for } |k|>1}$$

$$S_n \equiv \sum_{r=1}^{n}\frac{r^2}{k^r}=\frac{1^2}{k}+\frac{2^2}{k^2}+\frac{3^2}{k^3}+\frac{4^2}{k^4}+\ldots+\frac{n^2}{k^r}$$

$$S_n = \frac{1^2}{k^1}+\frac{2^2}{k^2}+\frac{3^2}{k^3}+\frac{4^2}{k^4}+\frac{5^2}{k^5}+\ldots+\frac{(n-1)^2}{k^{n-1}}+\frac{n^2}{k^n}$$

$$k\cdot S_n = \frac{1^2}{k^0}+\frac{2^2}{k^1}+\frac{3^2}{k^2}+\frac{4^2}{k^3}+\frac{5^2}{k^4}+\frac{6^2}{k^5}+\ldots+\frac{n^2}{k^{n-1}}$$

$$k\cdot S_n - S_n = \frac{1}{k^0}+\frac{2^2-1^2}{k^1}+\frac{3^2-2^2}{k^2}+\frac{4^2-3^2}{k^3}+\ldots+\frac{n^2-(n-1)^2}{k^{n-1}}-\frac{n^2}{k^n}$$

$$(k-1)\cdot S_n = 1+\frac{3}{k^1}+\frac{5}{k^2}+\frac{7}{k^3}+\frac{9}{k^4}+\frac{11}{k^5}+\ldots+\frac{2n-1}{k^{n-1}}-\frac{n^2}{k^n}$$

$$=\left(1-\frac{n^2}{k^n}\right)+\sum_{r=1}^{n-1}\frac{2r+1}{k^r}$$

$$=\left(1-\frac{n^2}{k^n}\right)+2\times\sum_{r=1}^{n-1}\frac{r}{k^r}+\sum_{r=1}^{n-1}\frac{1}{k^r}$$

$$=\left(1-\frac{n^2}{k^n}\right)+\left(\frac{2}{k-1}\right)\left(1-\frac{n-1}{k^{n-1}}\right)+2\left(\frac{1}{k-1}\right)^2\left(1-\frac{1}{k^{n-2}}\right)+\left(\frac{1}{k-1}\right)\left(1-\frac{1}{k^{n-1}}\right)$$

using 420F *and* 100F

$$S_n = \frac{1}{(k-1)}\left(1-\frac{n^2}{k^n}\right)+\frac{2}{(k-1)^2}\left(1-\frac{n-1}{k^{n-1}}\right)+\frac{2}{(k-1)^3}\left(1-\frac{1}{k^{n-2}}\right)+\frac{1}{(k-1)^2}\left(1-\frac{1}{k^{n-1}}\right)$$

$$\sum_{r=1}^{n}\frac{r^2}{2^r}=\left(1-\frac{n^2}{2^n}\right)+2\left(1-\frac{n-1}{2^{n-1}}\right)+2\left(1-\frac{1}{2^{n-2}}\right)+\left(1-\frac{1}{2^{n-1}}\right)$$

$$=6-\left(\frac{n^2}{2^n}+\frac{n}{2^{n-2}}-\frac{1}{2^{n-2}}+\frac{1}{2^{n-3}}+\frac{1}{2^{n-1}}\right)$$

$$=6-\left(\frac{n^2}{2^n}-\frac{1}{2^{n-1}}+\frac{n}{2^{n-2}}+\frac{1}{2^{n-3}}\right)$$

520FE

$$\boxed{\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\ldots+\frac{n^2}{2^r}=6-\left(\frac{n^2}{2^n}-\frac{1}{2^{n-1}}+\frac{n}{2^{n-2}}+\frac{1}{2^{n-3}}\right)}$$

520UE

$$\boxed{\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\frac{5^2}{2^5}+\ldots=\quad 6}$$

520AU
$$\boxed{1^2-\frac{2^2}{k}+\frac{3^2}{k^2}-\frac{4^2}{k^3}+\frac{5^2}{k^4}-\ldots=\quad\frac{k^2(k-1)}{(k+1)^3}}$$

$$S\equiv\sum_{r=0}^{\infty}\frac{(-1)^r(r+1)^2}{k^r}=1^2-\frac{2^2}{k}+\frac{3^2}{k^2}-\frac{4^2}{k^3}+\frac{5^2}{k^4}-\frac{6^2}{k^5}+\frac{7^2}{k^6}-\ldots$$

$$S=1^2-\quad\frac{2^2}{k}\quad+\quad\frac{3^2}{k^2}\quad-\quad\frac{4^2}{k^3}\quad+\quad\frac{5^2}{k^4}\quad-\quad\frac{6^2}{k^5}\quad+\ldots$$

$$\frac{S}{k}=\quad+\quad\frac{1^2}{k}\quad-\quad\frac{2^2}{k^2}\quad+\quad\frac{3^2}{k^3}\quad-\quad\frac{4^2}{k^4}\quad+\quad\frac{5^2}{k^5}\quad-\ldots$$

$$S+\frac{S}{k}=1\quad-\frac{2^2-1^2}{k}+\frac{3^2-2^2}{k^2}-\frac{4^2-3^2}{k^3}+\frac{5^2-4^2}{k^4}-\frac{6^2-5^2}{k^5}+\ldots$$

$$\left(1+\frac{1}{k}\right)S=1\quad-\quad\frac{3}{k}\quad+\quad\frac{5}{k^2}\quad-\quad\frac{7}{k^3}\quad+\quad\frac{9}{k^4}\quad-\quad\frac{11}{k^5}+\ldots$$

$$\left(\frac{k+1}{k}\right)S=\quad1\quad-\frac{2+1}{k}\quad+\quad\frac{4+1}{k^2}\quad-\quad\frac{6+1}{k^3}\quad+\quad\frac{8+1}{k^4}\quad-\quad\frac{10+1}{k^5}+\ldots$$

$$=1-\left(\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\ldots\right)-2\left(\frac{1}{k}-\frac{2}{k^2}+\frac{3}{k^3}-\frac{4}{k^4}+\frac{5}{k^5}-\ldots\right)$$

$$\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}-\frac{1}{k^4}+\frac{1}{k^5}-\ldots=\quad\frac{1}{k+1}$$ [050AU]

$$\frac{1}{k}-\frac{2}{k^2}+\frac{3}{k^3}-\frac{4}{k^4}+\ldots=\quad\frac{k}{(k+1)^2}$$ [430AU]

$$\left(\frac{k+1}{k}\right)S=\quad1-\frac{1}{k+1}-\frac{2k}{(k+1)^2}$$

$$S=\frac{k}{k+1}-\frac{k}{(k+1)^2}-\frac{2k^2}{(k+1)^3}$$

$$=\frac{k\left(k^2+2k+1\right)-k(k+1)-2k^2}{(k+1)^3}$$

$$=\frac{k^3-k^2}{(k+1)^3}\quad=\quad\frac{k^2(k-1)}{(k+1)^3}$$

520AUE
$$\boxed{1^2-\frac{2^2}{5}+\frac{3^2}{5^2}-\frac{4^2}{5^3}+\frac{5^2}{5^4}-\frac{6^2}{5^5}+\ldots=\quad\frac{5^2(5-1)}{(5+1)^3}=\frac{25}{54}}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r(r+1)^2}{5^r}=1^2-\frac{2^2}{5}+\frac{3^2}{5^2}-\frac{4^2}{5^3}+\frac{5^2}{5^4}-\frac{6^2}{5^5}+\frac{7^2}{5^6}-\ldots=\frac{25}{54}$$ J350

Series for π

There are many different series available to calculate π. Some converge faster than others, and therefore are better at producing higher precision results. In terms of the index variable *r* in the summations below, it is not fair to consider *r* as the sole metric of computational complexity. For example, combining two consecutive terms together reduces the *r*-index, but has not changed the convergence speed at all.

$$4\times\sum_{r=0}^{\infty}\frac{(-1)^r}{2r+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\ldots$$

J077

this is also $4\times\tan^{-1}(1)=\pi$ J734

$$2\sqrt{2}\times\sum_{r=0}^{\infty}(-1)^r\left(\frac{1}{2r+1}+\frac{1}{2r+3}\right)=\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots$$

J076

$$4-8\times\sum_{r=1}^{\infty}\frac{1}{(4r-1)(4r+1)}=4-8\times\left(\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\frac{1}{11\cdot 13}+\ldots\right)$$

_J077

$$4-8\times\sum_{r=1}^{\infty}\frac{1}{(16r^2-1)}=4-8\times\left(\frac{1}{15}+\frac{1}{63}+\frac{1}{144}+\ldots\right)$$

J077

$$2+16\times\sum_{r=0}^{\infty}\frac{1}{(4r+1)(4r+3)(4r+5)}=2+16\times\left(\frac{1}{1\cdot 3\cdot 5}+\frac{1}{5\cdot 7\cdot 9}+\frac{1}{9\cdot 11\cdot 13}+\ldots\right)$$

J238

$$\frac{8}{3}+8\times\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)(2r+3)(2r+5)}=\frac{8}{3}+8\times\left(\frac{1}{1\cdot 3\cdot 5}-\frac{1}{3\cdot 5\cdot 7}+\frac{1}{5\cdot 7\cdot 9}-\ldots\right)$$

J240

$$3+4\times\sum_{r=0}^{\infty}\frac{(-1)^{r+!}}{(2r)(2r+1)(2r+2)}=3+4\times\left(\frac{1}{2\cdot 3\cdot 4}-\frac{1}{4\cdot 5\cdot 6}+\frac{1}{6\cdot 7\cdot 8}-\ldots\right)$$

J244

$$\left(4\times\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)\cdot 2^{2r+1}}\right)+\left(4\times\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)\cdot 3^{2r+1}}\right)=$$

$$4\times\left(\frac{1}{2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\ldots\right)+4\times\left(\frac{1}{3}-\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}-\frac{1}{7\cdot 3^7}+\ldots\right)$$

ML

This is 1 of 4 two-term Machin-like formulae.[2]

600AU $\boxed{4\cdot\left(\frac{1}{2}-\frac{1}{3\cdot 2^3}+\frac{1}{5\cdot 2^5}-\frac{1}{7\cdot 2^7}+\ldots\right)+4\cdot\left(\frac{1}{3}-\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}-\frac{1}{7\cdot 3^7}+\ldots\right)=\pi}$

Write $\alpha=\tan^{-1}\left(\frac{1}{2}\right)$ so that $\tan(\alpha)=\frac{1}{2}$

$\beta=\tan^{-1}\left(\frac{1}{3}\right)$ so that $\tan(\beta)=\frac{1}{3}$

Use $\tan(\alpha+\beta)=\dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\cdot\tan(\beta)}=\dfrac{1/2+1/3}{1-(1/2)\cdot(1/3)}=\dfrac{5/6}{5/6}=1$

$$\tan^{-1}(1)=\alpha+\beta=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

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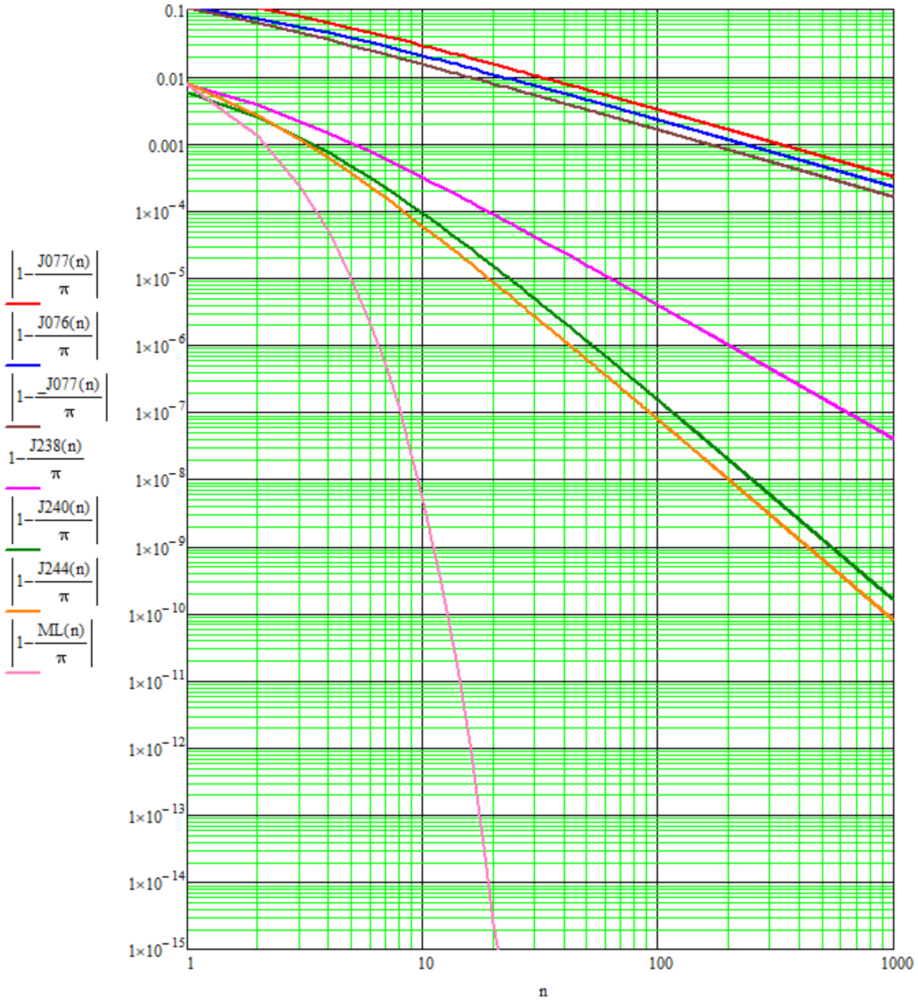
[2] https://mathworld.wolfram.com/Machin-LikeFormulas.html

Per-Unit Convergence Comparison



This next sum is famous as it was the solution to the Basel problem. It is also the value of the Euler zeta function with argument 2.

$$6\times\sum_{r=1}^{\infty}\frac{1}{r^2}=6\cdot\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots\right)=\pi^2 \qquad \text{J336 / P832}$$

The obvious thing to do is to add J336 and J337 to reduce the number of terms, using higher power terms more quickly. Sadly, this does not improve the convergence speed, J337 being the best.

$$8\times\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=8\cdot\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots\right)=\pi^2 \qquad \text{add / P835}$$

$$12\times\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2}=12\cdot\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\ldots\right)=\pi^2 \qquad \text{J337 / P833}$$

$$24\times\sum_{r=1}^{\infty}\frac{1}{(2r)^2}=24\cdot\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\ldots\right)=\pi^2 \qquad \text{P835}$$

$$9\times\left(1+\sum_{r=1}^{\infty}\frac{1}{(6r-1)^2}+\frac{1}{(6r+1)^2}\right)=9\cdot\left(1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\ldots\right)=\pi^2 \qquad \text{J338}$$

610U
$$\boxed{9\cdot\left(1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\ldots\right)=\pi^2}$$

Equality between Summations and Integrals

The origins of integral calculus stem from summation of the area under a continuous curve. Now we wish to reverse the process for the case of integer series summation.

Consider the summation:

$$S_n \equiv \sum_{r=1}^{n} r \equiv 1+2+3+4+5+6\ldots+n = \tfrac{1}{2}n(n+1)$$

The equivalent integral would appear to be:

$$I_n \equiv \int_1^n x\cdot dx = \left[\frac{x^2}{2}\right]_1^n = \tfrac{1}{2}\left(n^2-1\right)$$

Now whilst there is some similarity between the results, especially in the limit $n \to \infty$, the agreement is pretty poor. Perhaps changing the limits of integration would help:

$$I_n = \int_a^{n+b} x\cdot dx = \left[\frac{x^2}{2}\right]_a^{n+b} = \tfrac{1}{2}\left[(n+b)^2 - a^2\right] = \tfrac{1}{2}\left(n^2+2nb+b^2-a^2\right)$$

Equating the integral result with the known answer, it is evident that $a=b$ and $b=\tfrac{1}{2}$.

$$\sum_{r=1}^{n} r = \int_{0.5}^{n+0.5} x\cdot dx$$

700F $\boxed{1+2+3+4+5+6\ldots+n = \int_{0.5}^{n+0.5} x\cdot dx}$

$$S_n \equiv \sum_{r=1}^{n} r^2 \equiv 1^2+2^2+3^2+4^2+\ldots+n^2 = \tfrac{1}{6}\left(2n^3+3n^2+n\right)$$

$$I_n = \int_a^{n+b} x^2\cdot dx = \tfrac{1}{3}\left[x^3\right]_a^{n+b} = \tfrac{1}{3}\left((n+b)^3-a^3\right) = \tfrac{1}{6}\left(2n^3+6bn^2+6b^2n+2\left(b^3-a^3\right)\right)$$

Comparing terms we again have $a=b$. We now have two equalities to solve using one variable. Comparing coefficients of n^2 we have $3=6b$, whereas comparing coefficients of n we have $1=6b^2$. We apparently have a "choice" of $b=\frac{1}{2}$ or $b=\frac{1}{\sqrt{6}}$, although getting the coefficient of n^2 correct is better for larger *n*.

Whilst the integral is not exact, an error term can be used to get an equality:

$$\int_{0.5}^{n+0.5} x^2\cdot dx = \tfrac{1}{6}\left(2n^3+3n^2+1.5n\right) = \frac{n}{12} + \sum_{r=1}^{n} r^2$$

$$\sum_{r=1}^{n} r^2 = \int_{0.5}^{n+0.5}\left(x^2-\frac{1}{12}\right)\cdot dx$$

710F $\boxed{1^2+2^2+3^2+4^2+\ldots+n^2 = \int_{0.5}^{n+0.5}\left(x^2-\frac{1}{12}\right)\cdot dx}$

Version History

v1.10: Added 082F to 090FE. Renamed 010FE to 010F, then included a new 010FE. 1 November 2023.

v1.01: Corrected the last term in 150UE, 12 July 2023.

v1.00: First release, 26 July 2022.

http://lesliegreen.byethost3.com/publications.html