PI MU EPSILON JOURNAL

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PI MU EPSILON JOURNAL

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Problems for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

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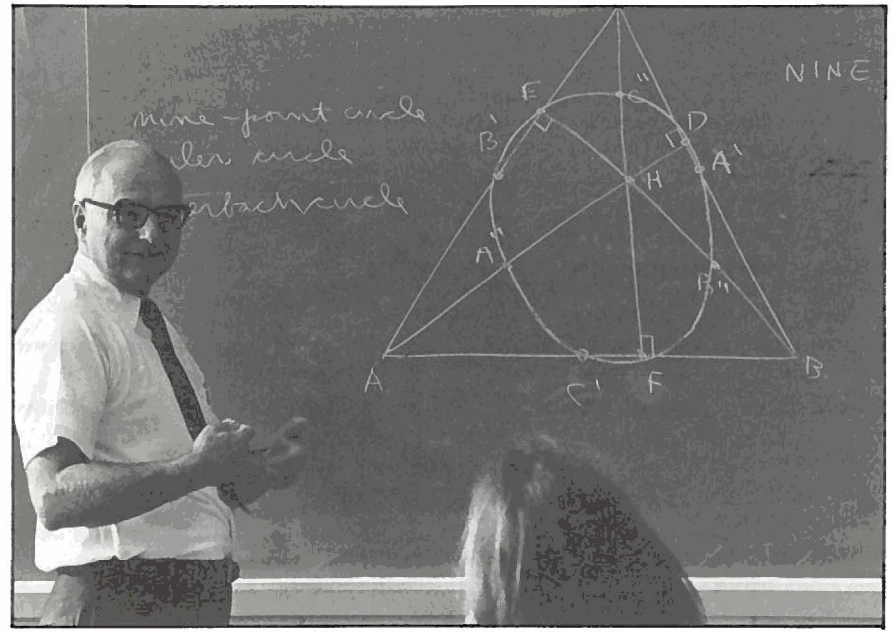


DEDICATION

This issue of the Pi Mu *Epsilon Journal* is dedicated to Joe Konhauser. Joseph D. E. Konhauser, Councillor for Pi Mu Epsilon, Editor of the Puzzle Section of the Pi Mu Epsilon Journal, and former Editor of the Journal, died on February 28, 1992, of complications following heart surgery. He was 67 years old. He is survived by his wife, Aileen, and his son, Dan.

Joe earned his bachelor's, master's, and doctorate degrees from Penn State University. From 1949 to 1955 he taught math at Penn State and was a mathematician at HRB–Singer Inc. in State College, PA. He was an associate professor of mathematics at the University of Minnesota for four years before joining the staff at Macalester College in St. Paul, MN, in 1968. He had retired from full time teaching at Macalester in 1991, but had returned to teach his popular geometry course this semester.

Besides his teaching and his wntributions to Pi Mu Epsilon, Joe had been a member of the committees that designed and evaluated tests for the USA Mathematical Olympiad and the William Lowell Putnam Mathematics Competition. He also had served as Fleviews Editor of the American Muthematical *Monthly*.

Joe had a real talent as a problem poser and solver. He had been Editor of the Puzzle Section since 1983; he had been creating Mathacrostics for the Puzzle Section since 1978. Perhaps even more remarkably, he had been posing a "Problem of the Week" at Macalester College for over 20 years, without repeating a problem.

Joe will be missed as a mathematician, as a renowned teacher, and as a friend.

PUZZLE SECTION

The Puzzle Section is for the enjoyment of those readers who are addicted to working doublecrostics or who find an occasional mathematical puzzle or word puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof.

COMMENTS ON PUZZLES 1–7, FALL 1991

For Puzzle **#1,** WILLIAM CHAU, RICHARD I. HESS, HENRY LIEBERMAN, and BOB **PRIELIPP** noted the following: Let K be the area of the triangle, s be its semi-perimeter, and r its inradius. Then $rs = K = 2s$, thus $r = 2$. Other relationships satisfied by these triangles were provided by MARK EVANS and CHARLES ASHBACHER.

The answer to Puzzle **#2,** the **"Bronzebach** Conjecture," is yes. Several decompositions were submitted. Perhaps the most concise was by BOB PRIELIPP:

If n is odd, $n = (n - 2) + 2$. If $n = 4k$, $n = (\frac{n}{2} - 1) + (\frac{n}{2} + 1)$.
If $n = 4k + 2$, $n = (\frac{n}{2} - 2) + (\frac{n}{2} + 2)$.

Solutions were submitted by CHARLES ASHBACHER, WILLIAM CHAU, VICTOR FESER, RICHARD I. HESS, HENRY LIEBERMAN, and DAVID SHOBE.

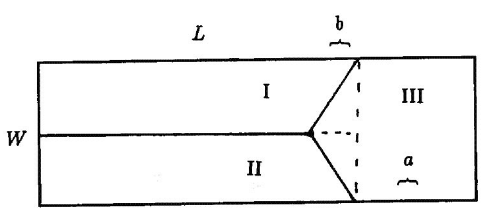
The first of the two solutions to Puzzle **#3 was** submitted hy DAVID SHOBE; the second solution appeared in ***The Oxford Guide to Word Games*** by Tony Augarde, 1984, **p.44.**

C	*I*	*R*	*C*	*L*	*E*
I	*N*	*U*	*R*	*E*	*S*
R	*U*	*D*	*E*	*S*	*T*
C	*R*	*E*	*A*	*S*	*E*
L	*E*	*S*	*S*	*O*	*R*
E	*S*	*T*	*E*	*R*	*S*

C	*I*	*R*	*C*	*L*	*E*
I	*C*	*A*	*R*	*U*	*S*
R	*A*	*R*	*E*	*S*	*T*
C	*R*	*E*	*A*	*T*	*E*
L	*U*	*S*	*T*	*R*	*E*
E	*S*	*T*	*E*	*E*	*M*

The solution to Puzzle # 4 is no. Suppose there were a solution with the set $\{7,8,9\}$. Consider the set containing 15. To complete the sum of 24, we need either 1 & 8 (but 8 is gone), or 2 & 7 (but 7 is gone), or 3 & 6, or 4 & 5. In either of these last two cases, there are no pairs of remaining numbers that will go with 14 to reach a sum of 24. (Solution by VICTOR FESER.) Others submitting solutions were CHARLES ASHBACHER, WILLIAM CHAU, **MARK** EVANS, RICHARD I. HESS, and HENRY LIEBERMAN.

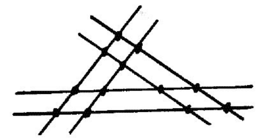
There were several different solutions to Puzzle # 5. The one that kept the three pieces the most similar in shape was submitted by MARK EVANS.



where $a = \frac{1}{3}L - \frac{1}{6}W$
$b = \frac{1}{3}W$

Others submitting solutions were WILLIAM CHAU, RICHARD I. HESS, DAVID SHOBE, and STAN WAGON.

For Puzzle #6, several solutions were submitted. The most common was



Submitting solutions were WILLIAM CHAU, MARK EVANS, VICTOR FESER, RICHARD I. HESS, and DAVID SHOBE.

For Puzzle #7, the resistance was found to be the Golden Ratio: $(\sqrt{5} - 1)/2$. Solvers were MARK EVANS, ROBERT GEBHARDT, HENRY LIEBERMAN, and DAVID SHOBE.

SOLUTION TO MATHACROSTIC NO. 33 (FALL 1991)

WORDS:

A.	openness	K.	entify	U.	Florentine enigma
B.	Verdict of Twelve	L.	left-handed	V.	theremin
C.	extenuate	M.	Ymir	W.	hypergraphics
D.	roses of grandi	N.	hem and haw	X.	even **steven**
E.	**Baily's** beads	O.	earth	Y.	corkscrew
F.	yatata yatata	P.	Alan Smithee	Z.	odd
G.	extent	Q.	right	a.	spread
H.	lute	R.	The Great White Way	b.	Modern Times
I.	on the **charts**	S.	switch	c.	Of Thee I Sing
J.	nines	T.	overtone	d.	spherical cow

AUTHOR AND TITLE: **OVERBYE** LONELY HEARTS OF THE COSMOS

QUOTATION: The veneer of existence was getting very, very thin, but it was in that last little crack of time – where space foamed into chaos and the spheres rang with harmonies undreamed of and symmetries were enfolded more intricately than a rose, where nothing happened and everything was possible – that the secret of gravity and existence lay.

SOLVERS: THOMAS F. BANCHOFF, Brown University, Providence, **RI;** JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; ROBERT C. GEBHARDT, Count College of Morris, Randolph, NJ; META HARRSEN, New Hope, PA; HENRY S. LIEBERMAN, Waban, MA; CHARLOTTE MAINES, **Rochester, NY;** STEPHANIE SLOYAN, Georgian Court College, Lakewood, NJ; JOHN L. VANIWAARDEN, Hope College, Holland, MI, and DONNA D. ASHBRIDGE, University of North Carolina – **Asheville,** NC; ALBERT WILANSKY, Lehigh University, Bethlehem, PA; and BARB ZEEBERG, Denver, CO.

MATHACROSTIC NO. 34

Proposed by Joseph D. E. Konhauser, shortly before his death.

The 223 letters to be entered in the numbered spaces in the grid will be identical to those in the 23 keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters on the Words will give the name of an author and the title of a book; the completed grid will be a quotation from that book. Solutions to Mathacrostic No. 34 should be sent to: Richard **Poss,** Pi Mu Epsilon **Journal,** St. Norbert College, De Pere, WI 54115.

DEFINITIONS	WORDS
A. a migratory Australian **cuckoo** (2 **wds.**)	35 48 107 114 161 192 67 200 23
B. popular **name** of **Dilworth's** 1740 "A **New Guide** to the English Tongue." **(comp)**	181 49 60 34 125 86 74 43 162
C. study of disease by **symptoms**	179 122 147 139 44 169 50 218 66
D. probability the first to give a theoretical construction for all the five regular solids and to show how to **inscribe then** in a sphere **(3 wds.)**	221 36 176 54 197 102 27 10 145 208 90 41 188 135 120 159 70 68
E. **game** plan	89 155 28 115 213 55 77 129 175
F. secret asset or ploy **(4 wds.)**	220 4 101 173 123 24 93 194 32 151 140 178
G. a **musical** means by which to identify characters, ideas, and objects as they occur in different situations and at different **times**	166 210 171 127 100 25 31 1 111
H. invented and patented by Kenneth **Snelson,** it has added a **new component** to the elegance and airiness	63 33 116 3 19 46 84 141 217 185
I. **must** starting point of the space **frame** for a cycle on a graph **(2 wds.)**	133 53 167 203 187 94 13 149 215 73
J. a fanciful product of the mind	158 85 198 65 134 112 51
K. a field of granular **snow**	126 9 214 137
L. **used** in prescriptions - of each an equal quantity	17 80 99
M. the **nickname** of the largest **simple** sporadic group (2 **wds.**)	121 21 96 38 144 83 180 204 6 223
N. to deprive of vital content or force	39 132 212 154 177 72 52 164 195 12
O. rural setting for **Schubert** opera **(2 wds.)**	182 57 47 69 124 104 113 211 216 165 108 130 37 76
P. incident	148 160 15 98 62 109 75
Q. leaving no **loophole**	191 56 170 20 5 88 64 30
R. marked vessel anchored at a chartered point to serve as an aid to navigation	190 82 142 103 106 16 128 153 61
S. **famous** or **unfamous** (3 **wds.**)	143 2 42 136 14 184 97 78 205 193 110 219 150 163
T. all out (3 wds.)	209 22 156 92 59 29 7 105 168 186 87 117
U. unpublished	118 91 199 71 183 157 207 152

V. **biotechnology**	79 138 119 196 58 40 18 202 8 172
W. originally **developed around 1968-70** in an **attempt to understand** the **strong** nuclear force; if successful **would provide** the —— le **unification** of physics (2 **wds.**)	146 81 206 189 45 26 174 11 131 222 95 201

1 G	2 S	■	3 H	4 F	5 Q	6 M	7 T	8 V	9 K	■	10 D	11 W	12 N
13 I	14 S	■	15 P	16 R	■	17 L	■	18 V	19 H	20 Q	21 M	22 T	23 A
■	24 F	25 G	■	26 W	27 D	28 E	■	29 T	30 Q	■	31 G	32 F	33 H
■	34 B	35 A	36 D	37 O	38 M	39 N	■	40 V	41 D	■	42 S	43 B	44 C
45 W	46 H	47 O	■	48 A	49 B	50 C	51 J	52 N	53 I	54 D	55 E	56 Q	57 O
58 V	■	59 T	60 B	61 R	62 P	63 H	64 Q	65 J	66 C	67 A	68 D	■	69 O
70 D	71 U	■	72 N	73 I	74 B	75 P	76 O	77 E	78 S	79 V	80 L	81 W	■
82 R	83 M	■	84 H	85 J	86 B	87 T	88 Q	89 E	90 D	91 U	■	92 T	93 F
94 I	95 W	96 M	■	97 S	98 P	■	99 L	■	100 G	101 F	102 D	103 R	104 O
105 T	■	106 R	107 A	108 O	■	109 P	110 S	111 G	112 J	113 O	114 A	■	115 E
116 H	117 T	118 U	119 V	120 D	121 M	122 C	123 F	124 O	125 B	126 K	127 G	■	128 R
129 E	130 O	131 W	132 N	133 I	134 J	■	135 D	136 S	137 K	■	138 V	139 C	140 F
141 H	142 R	143 S	144 M	145 D	146 W	■	147 C	148 P	149 I	150 S	151 F	152 U	■
153 R	154 N	■	155 E	156 T	157 U	■	158 J	159 D	160 P	161 A	162 B	163 S	164 N
165 O	166 G	167 I	■	168 T	169 C	170 Q	■	171 G	172 V	■	173 F	174 W	■
175 E	176 D	177 N	178 F	179 C	180 M	181 B	182 O	183 U	184 S	185 H	■	186 T	187 I
188 D	189 W	190 R	191 Q	192 A	193 S	194 F	■	195 N	196 V	■	197 D	198 J	199 U
200 A	201 W	■	202 V	203 I	204 M	205 S	206 W	207 U	208 D	209 T	210 G	211 O	■
212 N	213 E	214 K	215 I	216 O	217 H	218 C	219 S	220 F	221 D	222 W	223 N	■	■

PUZZLES FOR SOLUTION

To give some idea of the types of problem that Joe Konhauser liked to devise, this issue's Puzzle Section will present some of Joe's puzzles that had previously appeared in the *Journal.* The solution to each puzzle was discussed in the issue immediately following the puzzle's appearance.

1. *This problem first appeared in the Spring, 1983, issue of the* Journal.

In the square array

$$\begin{matrix} A & B & C \\ C & B & D \\ E & C & F \end{matrix}$$

each letter represents one of the digits 0 through 9. Determine the correspondence, given that:

(1) ABC and CBD are primes,

(2) BBC and CDF are perfect squares, and

(3) ACE and ECF are perfect cubes.

2. *(Fall, 1983)*

Sketch a graph (a finite collection of nodes and arcs) such that exactly three arcs terminate at each node and such that it is not possible to color the arcs with three colors so that no two arcs that are the same color terrninate at the same node.

3. *(Fall, 1983)*

The eight numbers {2, 3, 4, 6, 9, 14, 22, 31} have sum 91 and the property that taken two at a time the 28 sums obtained are all different. Are you able to find 8 positive integers with sum less than 91 with the same property?

4. *(Spring, 1984)*

Using just two colors, in how many distinguishable ways can one color the edges of a regular tetrahedron?

5. *(Fall, 1984)*

The trio of positive integers {5, 20, 44} has the property that the sum of any two of its members is a perfect square. Can you find a set of four distinct positive integers such that the sum of any three is a perfect square?

6. *(Spring, 1985)*

With a pair of compasses draw a circle on a plane. Then, without changing the opening of the compasses, draw a circle on a sufficiently large sphere. Which circle encloses the larger area?

7. *(Fall, 1987)*

Bored in a calculus class, a student started to play with a hand-held calculator. A four-digit number was entered, followed by the "square" key. To the surprise (and delight) of the student, the four terminal digits of the result were the same digits in the same order as those in the number which had been squared. What was that number?

THE RICHARD V. ANDREE AWARDS

Richard V. Andree, Professor Emeritus of the University of Oklahoma, died on May 8, 1987, at the age of 67. Professor Andree was a Past-President of Pi Mu Epsilon. He had also served the society as Secretary-General and as Editor of the Pi *Mu Epsilon Journal.* The Society Council has designated the prizes in the National Student Paper Competition as Richard V. Andree Awards.

First prize winner for 1991 is Amy Pinegar, for her paper "Inversions and Adjacent Transpositions," which appeared in the fall issue of the *Journal.* Amy prepared this paper, under the supervision of Dr. David Sutherland, while she was a senior at Middle Tennessee State University. She also presented the paper at the August, 1990, national Pi Mu Epsilon meeting at Columbus, Ohio. Amy will receive $200.

Second prize winner is Shannon Spittler, for her paper "A Math Problem Within an Antique Clock Label," which also appeared in the fall issue of the *Journal.* Shannon prepared this paper while she was a junior English major at Miami University in Ohio. She will receive $100.

Third prize winner is Judy Kenney, for her paper "Turning Triangles into Circles," which also appeared in the fall issue of the *Journal.* Judy prepared this paper while she was a senior at the College of St. Benedict. The problem was suggested to her by Dr. Steven Krantz while she was participating in an NSF Summer Research program at Washington University in St. Louis. Judy will receive $50.

There were three other student-written papers that appeared in 1991:

"Computerized Segmentation of Liver Structures from CT Images," by Heng Hak Ly, of Illinois Benedictine College. Heng prepared this paper with the help of Dr. Maryellen Giger and Dr. Rose Carney.

"A Note on a Paper of S. H. Friedberg," by Janet Valasek, of Penn State University – New Kensington Campus. Janet prepared this paper with the assistance of Dr. Javier Gomez-Calderon.

"A Pre-Calculus Method for Deriving Simpson's Rule," by John White, of Marshall University.

The current issue of the *Journal* contains two papers written by students:

"Change Ringing: Mathematical Music" was written by Heather DeSimone while she was a senior at Youngstown State University. She is currently attending graduate school at the College of William and Mary.

"Rings of Small Order" was written by Michael Lin while he was a senior at Moorhead High School, in Moorhead, MN. He is currently a freshman at Stanford University.

Joel Atkins, the winner of third prize in the 1990 Competition, wishes to acknowledge the guidance of Dr. Jack Kinney of Rose-Hulman Institute of Technology.

RINGS OF SMALL ORDER

Michael H. Lin
Stanford University

Introduction

Since all finite **abelian** groups have a simple structure, a straightforward way to find all finite rings is to begin with its additive group. If we are given one particular additive group, say $G, +$, to work with, the problem is reduced to finding all binary operations "$\cdot$" on G that are associative and that are left and right diitributive over "+". This will largely be a matter of trial and error, and thus in general will be computation-intensive. One naive approach would be to try **all** n^{n^2} **possible** multiplication tables and to check associativity and distributivity for each one of them.

In this paper, a more efficient approach is developed, and a computer program implementing it was written for use on an IBM PC compatible. This program takes as input a standard decomposition of the additive group, and outputs the multiplication tables of **all possible** rings with that **additive group.** The program does not determine which outputs are isomorphic. It works for any additive group with order up to **127,** although in many cases a complete run would be impractical because of both the amount of generated output and the length of run time.

Notation

The order of any group G will be denoted by $|G|$; likewise, the order of any element g will be denoted by $|g|$. The cyclic group of order n will be denoted by C_n.

Let our given additive group G of order n be expressed as a direct product of nontrivial cyclic groups $H_1 \times H_2 \times \ldots \times H_r$, where $|H_k|$ divides $|H_{k-1}|$ for $1 < k \leq r$. (Such a representation is uniquely determined by G.)

For $1 \leq k \leq r$, pick h_k in G such that

$$(h_k) = \{0\} \times \ldots \times \{0\} \times H_k \times \{0\} \times \ldots \times \{0\}.$$

Let $B = \{h_1, h_2, \ldots, h_r\}$, so that $(B) = G$.

Algorithm

1. Input the orders of the H_k.
2. Compute the addition table and other information about G (such as the multiples and order of each element).
3. Set up a loop so that each passage through the loop assigns a value in G to each of the r^2 products obtained from B. Successive passages through the loop assign every possible value in G to each of the r^2 products.
4. Check the necessary condition that $|h_j|(h_j \cdot h_k) = |h_k|(h_j \cdot h_k) = 0 \quad \forall h_j, h_k \in B$.
5. Define the remaining products within G by distributivity: The distributive properties

$$a(b+c) = ab + ac \quad \text{and} \quad (a+b)c = ac + bd$$

can be restated as

$$\left(\sum x_j\right)\left(\sum y_k\right) = \sum\sum x_j y_k.$$

So for any $x = \sum a_j h_j$, $y = \sum b_k h_k$, distributivity gives

$$\begin{aligned} x \cdot y &= \left(\sum a_j h_j\right) \cdot \left(\sum b_k h_k\right) \\ &= \sum\sum (a_j h_j \cdot b_k h_k) \\ &= \sum\sum a_j b_k (h_j \cdot h_k). \end{aligned}$$

Using the condition of Step 4, it can be shown that the operation "$\cdot$" as given here is well-defined.

6. Check for associativity within B; i.e., that

$$(h_i \cdot h_j) \cdot h_k = h_i \cdot (h_j \cdot h_k) \quad \forall h_i, h_j, h_k \in B. \qquad (*)$$

This is sufficient because, if (*) is satisfied,

$$\begin{aligned} \left(\sum a_i h_i \cdot \sum b_j h_j\right) \cdot \sum c_k h_k &= \left[\sum\sum a_i b_j (h_i \cdot h_j)\right] \cdot \sum c_k h_k \\ &= \sum\sum\sum a_i b_j c_k [(h_i \cdot h_j) \cdot h_k] \\ &= \sum\sum\sum a_i b_j c_k [h_i \cdot (h_j \cdot h_k)] \\ &= \sum a_i h_i \cdot \left(\sum b_j h_j \cdot \sum c_k h_k\right). \end{aligned}$$

7. If Steps 4 and 6 are both satisfied, we have generated a ring. Output it.
8. **Repeat** Steps 4 through 7 as indicated in Step 3.

Results

The computer program was written in IBM PC assembly language. In addition to the multiplication tables for each ring and the total number of rings generated, the program also outputs the first $n-1$ **powers** of each element for each ring. This shows certain properties of the ring at a glance - for example, how many squares are non-zero, and whether all cubes are zero - and thus makes it easier to see which rings might be isomorphic. It also tells at a glance that some pairs of rings are not isomorphic.

The generated rings for some additive groups were hand-classified according to isomorphism. The obtained results agreed with the list published in [1] of all 24 rings, up to isomorphism, of order less that 8.

The data in the following table were obtained using an 8Mhz IBM PC/XT clone running a stripped-down version of the computer program. The deleted parts of the program were those that computed the multiples and powers of each element. It should be noted that this trimmed version is significantly faster than the original program.

For cyclic additive groups of order n, the program produced a total of **n** rings. It was proved in [2] that the number of non-isomorphic rings with additive group C_n is the number of divisors of n; this was verified for n up to **10.**

Structure of Additive Group	Total Number of Rings Produced	Computation Time
$C_2 \times C_2$	28 (8 non-isomorphic)	~ 0.3 sec.
$C_4 \times C_2$	60 (20 non-isomorphic)	1 sec.
$C_6 \times C_2$	84 (16 non-isomorphic)	3 sec.
$C_8 \times C_2$	120	7 sec.
$C_{10} \times C_2$	140	14 sec.
$C_{12} \times C_2$	180	25 sec.
$C_{14} \times C_2$	196	39 sec.
$C_3 \times C_3$	121 (8 non-isomorphic)	17 sec.
$C_6 \times C_3$	242	116 sec.
$C_9 \times C_3$	405	6.2 min.
$C_{12} \times C_3$	484	15.0 min.
$C_{15} \times C_3$	605	28.9 min.
$C_4 \times C_4$	616	7.8 min.
$C_8 \times C_4$	1376	57 min.
$C_{12} \times C_4$	1848	196 min.
$C_{16} \times C_4$	2816	7.6 hr.
$C_{20} \times C_4$	3080	14.8 hr.
$C_5 \times C_5$	793	106 min.
$C_{10} \times C_5$	1586	14.1 hr.

The above data suggest that if s and t are relatively prime, the number of rings produced for additive group $C_{st} \times C_s$ is t times the number for $C_s \times C_s$.

For large n and small r (as defined in Notation), Step 5, i.e., the completion of the multiplication table, dominates the other steps in terms of the computation time needed. Also, the time required for one execution of Step 5 is approximately proportional to the size of the multiplication table. Thus, for large n and small r, a rough indicator of the total computation time would be

(the number of potential rings that pass Step 4) $\times$ (n^2).

For an additive group of the form $C_{st} \times C_s$, this expression simplifies to $s^{12}t^3$.

Additional Observations

While the rings with additive group $C_4 \times C_2$ were being hand-classified up to isomorphism, it was noticed that four rings were anti-automorphic; i.e., that there existed a bijection f on each of the rings such that $f(x \cdot y) = f(y) \cdot f(x)$ for all x and y in the ring. The following theorems were then formulated.

Preliminaries

Let $A_s,+$ be the abelian group $C_{s^2} \times C_s = (a) \times (b)$. We define $f : A_s \longrightarrow A_s$ by $f(pa+qb) = p(a-b)+q(-b)$ for all $p, q \in \mathbb{Z}$. It can be shown that f is its own inverse; so f is a bijection. Also, it follows immediately from the definition that f preserves the operation "+".

Theorem 1. Let R_s be the ring with additive group A_s and with multiplication defined by the relations

$$a \cdot a = a \cdot b = 0,$$
$$b \cdot a = b \cdot b = sa.$$

Then R_s is a non-commutative ring that is anti-isomorphic to itself.

Proof. It can be verified that

$$|x|(x \cdot y) = |y|(x \cdot y) = 0 \quad \forall x, y \in \{a, b\},$$

and that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in \{a, b\}.$$

It follows from Steps 5 and 6 that R_s is a ring. Also, R_s is obviously non-commutative. By the preliminaries, f is a bijection and $f(x+y) = f(y) + f(x)$ $\forall x, y \in R_s$. We shall show that $f(x \cdot y) = f(y) \cdot f(x)$ $\forall x, y \in R_s$.

$$f(a \cdot a) = f(0) = 0;$$
$$f(a) \cdot f(a) = (a-b) \cdot (a-b) = a \cdot a - a \cdot b - b \cdot a + b \cdot b = 0 - 0 - sa + sa = 0.$$

$$f(a \cdot b) = f(0) = 0;$$
$$f(b) \cdot f(a) = -b \cdot (a-b) = -b \cdot a + b \cdot b = -sa + sa = 0.$$

$$f(b \cdot a) = f(sa) = sf(a) = s(a-b) = sa - sb = sa - 0 = sa;$$
$$f(a) \cdot f(b) = (a-b) \cdot (-b) = -a \cdot b + b \cdot b = 0 + sa = sa.$$

$$f(b \cdot b) = f(sa) = \ldots = sa;$$
$$f(b) \cdot f(b) = (-b) \cdot (-b) = b \cdot b = sa.$$

The general fact that $f(x \cdot y) = f(y) \cdot f(x)$ now follows by distributivity.

Therefore, f is an anti-automorphism of R_s.

Theorem 2. Let Q_s be the ring with additive group A_s and with multiplication defined by the relations

$$a \cdot b = 0,$$
$$a \cdot a = b \cdot a = b \cdot b = sa.$$

Then Q_s is a non-commutative ring that is anti-isomorphic to itself.

The proof is exactly as for Theorem 1, except

$$f(a \cdot a) = f(sa) = \ldots = sa;$$
$$f(a) \cdot f(a) = (a-b) \cdot (a-b) = a \cdot a - a \cdot b - b \cdot a + b \cdot b = sa - 0 - sa + sa = sa.$$

These two theorems raise some interesting questions about anti-automorphic non-commutative rings: What conditions upon an additive group are necessary and sufficient for there to exist anti-automorphic non-commutative rings with this additive group? How many anti-automorphic non-commutative rings exist, up to isomorphism, for any given additive group? Can a general description of their multiplication tables be given? What can be said about their structure? What other properties do they have?

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A computer program listing and a sample of the program output can be obtained by writing the author at P.O. Box 4048 Stanford, CA 94309 (e-mail: michelin@leland.stanford.edu).

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The ***Pi Mu Epsilon Journal*** was founded in **1949** and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements, and contributions to the Puzzle Section and Problem Department of the ***Journal*** should be directed toward this group.

Undergraduates and beginning graduate students are urged to submit papers to the ***Journal*** for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. Some guidelines are:

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5. Papers should not exceed **10** pages in length.
6. Figures provided by the author should be camera-ready.
7. Papers should be submitted in duplicate to the Editor.

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CHANGE RINGING: MATHEMATICAL MUSIC

Heather DeSimone
Youngstown State University

1. Introduction

Before the eighth century most church bells were small and rung by hand. These bells were made of iron and did not have good tone quality. Making bells from different alloys began around the eighth century. Using bronze, it became possible to create much bigger and louder bells. It was also discovered that different tones could be made by varying the thickness of the bell wall and the composition of the bell metal. The size of the bell also affected its sound. For example, the bigger bells made deeper sounding notes. At this time, large bells were being installed in church towers all over Europe. At the turn of the thirteenth century a gradual change in the shape of bells took place. The sides became longer and more concave, which improved tone.

As bells became larger and heavier, they became more difficult to ring. Consequently, methods of ringing evolved that did not require shifting the full weight of the bell. One of the methods, which is still employed today, is to swing a bell by a rope attached to the top until it is almost upside-down and then swing it back to complete the other half of the swing. This method was refined by mounting the bell on a half-wheel. A rope was then run around it and down to the floor, which provided a "stay" on the wheel's rim thereby preventing the bell from swinging all the way over. A final improvement was implemented soon after the Reformation when a whole wheel was introduced, allowing complete control over the bell. This improvement not only enabled the bell to stay in an upright position for as long as was needed, but more importantly, it allowed control over the speed of the swing. Pulling harder on the rope as it lowered sped up the swing. Conversely, retarding the rope as the bell swung up slowed down the swing. It was found that if two ringers of two different bells carried out these moves, the bells would change place in their order of ringing. 'It was this discovery, when applied to a number of bells, that made 'change-ringing' possible; and this is the foundation on which the whole art of bellringing is **based"(Camp, 15).**

2. Change Ringing

The basic strategy of change ringing is:

(1) to ring a given set of **bells** in all possible sequences;

(2) to move in a methodical fashion from one sequence to another; and

(3) to avoid repeating any sequence.

There are n! possible sequences for n bells. Each number of bells has a specific name as listed in the table.

Number of Bells	Name	Number of Changes
4	Singles	**24**
5	Doubles	**120**
6	Minor	**720**
7	Triples	**5,040**
8	Major	**40,320**
9	Caters	**362,880**
10	Royal	**3,628,800**
11	Cinques	**39,916,800**
12	**Maximus**	**479,001,600**

The object of change ringing is to produce all of the permutations on a set of bells **according** to a set of rules. The highest bell is called the treble bell and the lowest, the tenor. When they ring in descending order, from treble to tenor, they are said to be in rounds. The rules the **bellringers** must satisfy **are:**

(i) the peal must begin and end in rounds;

(ii) no bell may move more than one position from one change to the next; and

(iii) no bell may occupy the same position for more than two successive changes.

The last rule is sometimes relaxed.

The six changes on three bells can be rung as follows:

123
213
231
321
312
132
123

or in the reverse order, but only these two ways follow the rules. These bells follow a hunting course. This means each bell works by steps of one to the right or left until the bell is first or last in the change. The first bell moves from the first position, to the second position, and to the third position. The bell then stays in the third position for two consecutive changes before it moves back to the second position. It then moves to the first position and stays there for the last two changes.

Look at the first transition. It can be denoted by the transposition *(12)* meaning that the bells in position *1* and *2* change places. The two operators applied in the changes on the three bells alternately are $A = (12)$ and $B = (23)$. These generate the entire group of order six. Algebraically, the six changes on three bells can be represented as $(AB)^3$ because *A* and then *B* are applied three times.

With four bells this is a little more complicated.

	1234	*1342*	**1423**	**1234**
Plain Bob	*2143*	*3124*	**4132**	
Method	*2413*	*3214*	**4312**	
	4231	*2341*	**3421**	
	4321	*2431*	**3241**	
	3412	*4213*	**2314**	
	3142	*4123*	**2134**	
	1324	*1432*	**1243**	

These bells also follow a hunting course. In the beginning, bell 1 is moved one position to the right. It then stays in the last position for two changes before moving backwards to its original position. The other three bells follow a similar hunting pattern. As four bells hunt, they create eight changes. In general, if n bells hunt, the hunting generates a group of order $2n$. The process of hunting on four bells consists of alternately applying the two operators $A = (12)(34)$ and $B = (23)$. As stated before, these generate the first eight changes. Continuing to use these operators, specifically using *B*, would make the next sequence *1234*. This is commonly known to bellringers as "replacing rounds!' It is not desirable because all of the possible changes would not have been rung. In order to prevent this, and to continue, we employ the irregular move $C = (34)$. The second eight elements are generated by again applying *A* and *B*. After the irregular move *C* is applied again, the third eight elements are generated the same as the first two sets. The Plain Bob method can be algebraically symbolized with the operators as:

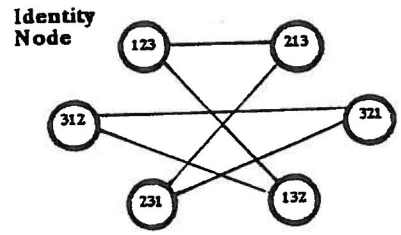
$$((AB)^3(AC))^3.$$

This notation means operators *A* and B are alternately applied three times.

Then *A* and *C* are applied. This complete pattern is repeated three times.

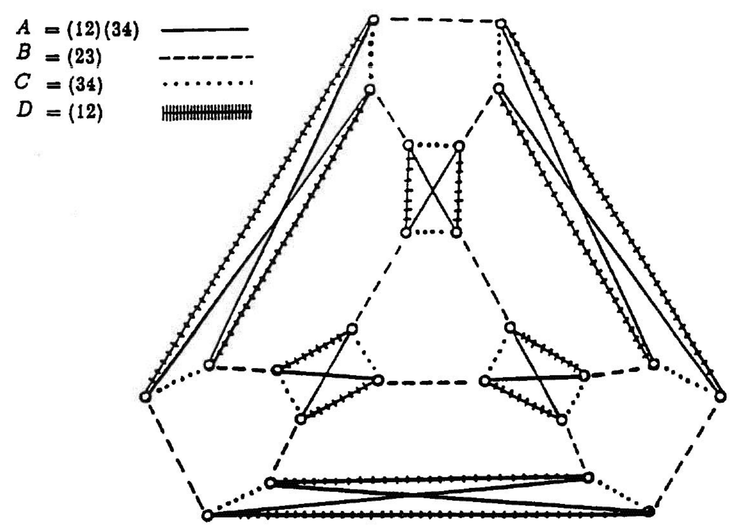
3. **Hamiltonian** Circuits

Any particular set of complete changes can be graphically represented as a Hamiltonian cycle. Let the nodes of the graph symbolize each change, that is, an ordering in which the bells are rung, and the edges connect the possible consecutive changes. Graphing the changes on three bells, the Hamilton circuit is easily found. It is also easy to see that this is the only one since there **are** no edges left out of the circuit. This shows that there are only two ways of ringing the changes dependingon which direction the identity node is exited.



In general, the number of nodes is equal to $n!$, where n is the number of bells. The number of edges going to or coming from one node depends on the number of possible changes. In the example of three bells, we can interchange bells *1* and *2*, *(12)* or bells *2* and *3*, *(23)*. These are the only two possibilities; therefore, there are two edges per node.

By increasing the number of bells by one, the number of nodes increases, as does the number of permutations. There are four possible ways to change from one sequence to another. The first three, which were discussed earlier, are **(12)(34)**, **(23)**, and *(34)*. The last is switching only the first two, *(12)*. So every one of the *24* nodes has four edges or is connected to four different nodes. This graph is more complicated than the one for three bells. A Hamiltonian circuit is not easy to find in the maze of **48** edges and *24* vertices; however, several can be found. The set of sequences discussed earlier is one example. The figure uses the form given in White [9].



This method is the most commonly rung and the most commonly displayed mathematically; however, there are others.

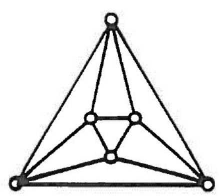
Names	Algebraic Description
Plain Bob	$((AB)^3AC)^3$
Reverse Bob	$(ABAD(AB)^2)^3$
Double Bob	$(ABADABAC)^3$
Canterbury	$(ABCDCBAB)^3$
Reverse Canterbury	$(DB(AB)^2DC)^3$
Double Canterbury	$(DBCDCBDC)^3$
Single Court	$(DB(AB)^2DB)^3$
Reverse Court	$(AB(CB)^2AB)^3$
Double Court	$(DB(CB)^2DB)^3$
St. Nicholas	$(DBADABDC)^3$
Reverse St. Nicholas	$(ABCDCBAC)^3$

It is worth noting that only the first three of these methods satisfy all three rules listed in section *2.* The remaining methods fail condition (iii) that states no bell may stay in the same position for more than two consecutive changes.

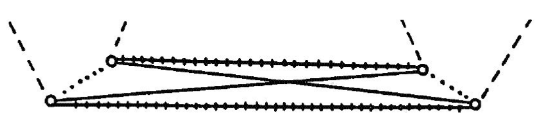
Using the given operators, two original sequences will be demonstrated. Alternating D and B with every sixth change using the A transition completes the necessary Hamiltonian cycle. Algebraically, this is represented as $((DB)^2(DA))^4$. And using the similar pattern $((CB)^2(CA))^4$ also produces the circuit. The patterns are alike in that the second set replaces the D's of the first set with C's.

All methods with four bells use exactly *24* of the *48* edges to complete the Hamiltonian circuit. So it seems possible to find two independent cycles on the same graph. None of the above examples are independent of each other. In other words, two completely different Hamiltonian circuits can not be found with the previous instances. Therefore, starting with *48* edges and completing a Hamiltonian circuit in *24* edges does not necessarily mean there are two totally independent Hamiltonian circuits on that graph, even though there are *24* unused edges.

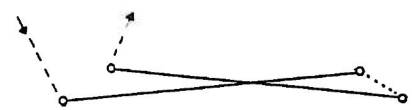
We will now show that such examples exist. By breaking down the earlier diagram into a simpler form where only the B connectors are left and each group of other lines are thought of as separate entities, we have:



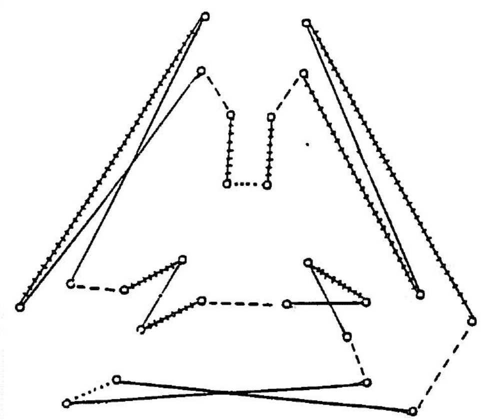
From this diagram two independent Hamiltonian circuits can easily be determined. Studying further, we find that there are exactly six different pairs. Also, notice that this graph and the other five are not symmetric. All of the previous examples are symmetric. Now that we know what path to follow going in and out of each vertex, we have to look at the vertices which represent the separate entities. For example, the bottom vertex looks like this:

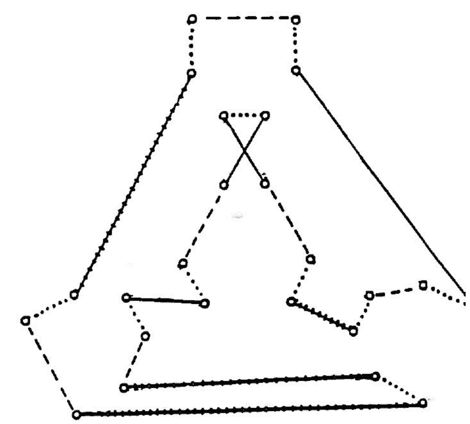


There are two possibilities to complete the Hamiltonian circuit.



All six vertices are similar, so there are 2^6 possibilities. Multiplying the 2^6 ways times the 6 ways from the B connectors, we have the *384* possibilities for two totally independent hamiltonian circuits on four bells. None of these graphs can be symmetric since the graph of B connectors is not symmetric. Notice this in the following example of two independent hamiltonian cycles.





Without symmetry, a pattern in the letters can not be found, and the changes can not be represented in a short algebraic form like the ones given earlier.

Going on to five bells causes even a bigger problem. The graph of the possibilities, alone, is complicated. There are seven possible changes from sequence to sequence: (12), (34), (45), (23), (12)(34), (12)(45), and (23)(45). So each of the *120* vertices is connected to seven other vertices. That is a total of 420 edges. It takes *120* of these edges to complete a Hamiltonian circuit. Since *420* is not a multiple of *120,* independent Hamiltonian circuits can not be found that use all the edges.

For six bells there are *720* changes, *12* transitions, and *4320* edges on the graph. Because there is an even number of transitions, the number of edges is a multiple of the number of changes. So it seems likely that independent Hamiltonian circuits that use all of the edges exist. Since there is no easy method for determining which graphs are Hamiltonian and each graph must be considered individually, determining whether there exists more than one Hamiltonian circuit on one graph can not be found using a theorem. Therefore, finding a method to find the Hamiltonian circuits is part of the problem.

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ELEMENTARY SYMMETRIC POLYNOMIALS, AN INTUITIVE APPROACH WITH APPLICATIONS TO COLLEGE ALGEBRA AND BEYOND

Daniel Replogle

This article will present an intuitive introduction to the elementary symmetric polynomials and describe some of their uses. It is written with the good college algebra student in mind. Everything it contains should be accessible to the student who has mastered college algebra.

Elementary symmetric polynomials are used frequently in advanced courses in algebra, and are not usually presented until then. However, seeing them earlier might give undergraduates a greater sense of the structure of algebra. Just as journals frequently use methods from advanced calculus to throw light upon topics from standard calculus courses, so topics from advanced algebra may sometimes be used to throw light upon topics from earlier algebra courses. An early introduction to elementary symmetric polynomials is seeing how each term in a polynomial depends upon the roots of that polynomial. This dependence will be shown and stated as a theorem, though no proof of this result will be given.

Consider the following products:

(1) $$(x-a)(x-b) = x^2-(a+b)x+ab.$$

(2) $$\begin{aligned}(x-a)(x-b)(x-c) &= (x^2-(a+b)x+ab)(x-c)\\ &= x^3-(a+b)x^2+abx-cx^2+c(a+b)x-abc\\ &= x^3-(a+b+c)x^2+(ab+ac+bc)x-abc.\end{aligned}$$

(3) $$\begin{aligned}(x-a)(x-b)(x-c)(x-d) &= [x^3-(a+b+c)x^2+(ab+ac+bc)x-abc](x-d)\\ &= x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2\\ &\quad -(abc+abd+acd+bcd)x+abcd.\end{aligned}$$

(4) $$\begin{aligned}(x-a)(x-b)(x-c)(x-d)(x-e) &= x^5-(a+b+c+d+e)x^4\\ &\quad +(ab+ac+ad+ae+bc+bd+be+cd+ce+de)x^3\\ &\quad -(abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)x^2\\ &\quad +(abcd+abce+abde+acde+bcde)x-abcde.\end{aligned}$$

Noting the above pattern, let $a_1, a_2, a_3, a_4, \ldots, a_k$ be the zeros of a monic polynomial (a polynomial where the coefficient of the highest degree term is 1) of degree k. Then define:

$$s_{1k} = a_1+a_2+a_3+\ldots+a_k$$

$$s_{2k} = a_1a_2+a_1a_3+\ldots+a_1a_k+a_2a_3+\ldots+a_2a_k+a_3a_4+\ldots+a_{k-1}a_k$$

$$\begin{aligned}s_{3k} &= a_1a_2a_3+a_1a_2a_4+\ldots+a_1a_2a_k+a_1a_3a_4+\ldots+a_1a_3a_k+a_1a_{k-1}a_k+\\ &\quad \ldots+a_2a_3a_4+\ldots+a_{k-2}a_{k-1}a_k\end{aligned}$$

$$s_{4k} = a_1a_2a_3a_4+\ldots+a_{k-3}a_{k-2}a_{k-1}a_k$$

$$s_{5k} = a_1a_2a_3a_4a_5+\ldots+a_{k-4}a_{k-3}a_{k-2}a_{k-1}a_k$$

$$s_{kk} = a_1a_2a_3a_4\ldots a_{k-2}a_{k-1}a_k.$$

The preceding definea the elementary symmetric polynomials on k **letters.*** **The** following might be helpful to keep in mind:

s_{1k} **is** the sum of all of the zeros,

s_{2k} is the sum of all of the disjoint **pairwise** product8 of zeros,

s_{3k} is the sum of all of the disjoint **3-wise** products of zeros, etc.

With all of **this** in mind and recalling the pattern observed above, we have the following theorem (which can be proved rigorously, for those who desire to do so):

Theorem: The monic polynomial $p(x)$ of degree k, with zeros $a_1, a_2, a_3, \ldots, a_k$ **is** given by:

$$p(x) = x^k - s_{1k}x^{k-1} + x_{2k}x^{k-2} - s_{3k}x^{k-3} + \ldots + (-1)^r s_{rk}x^{k-r} + \ldots \pm s_{k-1,k}x \mp s_{kk}.$$

Applications:

1. Find the monic polynomial $p(x)$ with zeros $1, 2, \sqrt{2}, -\sqrt{2}$.

$$s_{14} = 1 + 2 + \sqrt{2} - \sqrt{2} = 3.$$

$$\begin{aligned} s_{24} &= 1(2) + 1(\sqrt{2}) + 1(-\sqrt{2}) + 2(\sqrt{2}) + 2(-\sqrt{2}) + (\sqrt{2})(-\sqrt{2}) \\ &= 2 + \sqrt{2} - \sqrt{2} + 2\sqrt{2} - 2\sqrt{2} - 2 = 0. \end{aligned}$$

$$\begin{aligned} s_{34} &= 1(2)(\sqrt{2}) + 1(2)(-\sqrt{2}) + 1(\sqrt{2})(-\sqrt{2}) + 2(\sqrt{2})(-\sqrt{2}) \\ &= 2\sqrt{2} - 2\sqrt{2} - 2 - 4 = -6. \end{aligned}$$

$$s_{44} = 1(2)(\sqrt{2})(-\sqrt{2}) = -4.$$

So, $p(x) = x^4 - 3z^3 + 0x^2 - (-6)x + (-4) = x^4 - 3z^3 + 6x - 4$.

2. Find the polynomial $p(x)$ having zeros $i\sqrt{2}, -i\sqrt{2}$, and 2 with $p(3) = 2$.

$$s_{13} = 2 + i\sqrt{2} - i\sqrt{2} = 2.$$

$$\begin{aligned} s_{23} &= i\sqrt{2}(-i\sqrt{2}) + (i\sqrt{2})(2) - i\sqrt{2}(2) \\ &= 2 + 2i\sqrt{2} - 2i\sqrt{2} = 2. \end{aligned}$$

$$s_{33} = i\sqrt{2}(-1\sqrt{2})(2) = 4.$$

so, $p(x) = r(x^3 - 2x^2 + 2x - 4) = rx^3 - 2rx^2 + 2rx - 4r$.

$p(3)$ is given by:

$$\begin{array}{c|cccc} 3 & r & -2r & 2r & -4r \\ & & 3r & 3r & 15r \\ \hline & r & r & 5r & 11r \end{array}$$

So, $p(3) = llr$. But $p(3) = 2$, making $r = 2/11$. Thus

$$p(x) = \frac{2}{11}x^3 - \frac{4}{11}x^2 + \frac{4}{11}x - \frac{8}{11}.$$

3. Show that if p is a prime number, $p > 2$, then the sum of the pth roots of unity is zero, and their product is one.

Each of the pth roots of unity satisfies the equation $x^p - 1 = 0$. **Further,** this polynomial equation has p **roots.** So, if $a_1, a_2, \ldots, a_p$ are the roots of $x^p - 1 = 0$, then

$$\begin{aligned} (x - a_1)(x - a_2)\ldots(x - a_p) &= x^p - s_{1p}\,x^{p-1} + s_{2p}\,x^{p-2} - \ldots + s_{p-1,p}\,x - s_{pp} \\ &= x^p - 1. \end{aligned}$$

(We can be definite about the choice of signs because a prime > 2 is necessarily odd.) It follows that $s_{1p} = 0$ and $-s_{pp} = -1$. Thus $s_{1p} = 0$, so the sum of the pth roots of unity **is** zero (where p **is** a prime > 2).

Also $s_{pp} = 1$, so the product of the pth roots of unity (p a prime > 2) is one.

Additional Comments:

The usual method for finding the polynomials in Examples 1 and 2 might be quicker and simpler, but it will not help one to solve problems like that in Example 3. Also, to me, the usual method **seems** to be just a bit too tedious and it fails to reveal any structure. For comparison, here is how Example 1 is usually solved:

1a. Find the monic polynomial $p(x)$ with zeros $1, 2, \sqrt{2}, -\sqrt{2}$.

$$\begin{aligned} p(x) &= (x - 1)(x - 2)(x - \sqrt{2})(x - (-\sqrt{2})) \\ &= (x - 1)(x - 2)(x - \sqrt{2})(x + \sqrt{2}) \\ &= (x^2 - 3x + 2)(x^2 - 2) \\ &= x^4 - 3x^3 + 6x - 4. \end{aligned}$$

A careful look at this and Example 1 above, **I** think, reveals that the method using symmetric polynomials is somewhat less tedious and reveals more structure.

*Daniel **Replogle** prepared this **article shortly** after completing his master's at St. Louis University. **He** is **currently** a **graduate** student at the State **University of New York at** Albany.*

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SOLUTIONS TO ANTIDERIVATIVES USING A HYPERBOLIC FUNCTIONAL TRANSFORMATION

Timothy Holland
Alabama State University and St. Jude High School, Montgomery, AL

Most textbooks for elementary integral calculus include a section titled 'miscellaneous substitutions.' Among the types of problems that these sections generally consider are thcae which involve finding the antiderivatives of rational functions of sin(@)and $\cos(\beta)$. The traditional method of **solving** some of these problems uses the following substitution [1]:

$$\sin(\beta)=\frac{2x}{1+x^2},\quad \cos(\beta)=\frac{1-x^2}{1+x^2},\quad \text{and}\quad d\beta=\frac{2}{1+x^2}\,dx.$$

However, the use of the exponential and hyperbolic functions offers an alternative method for solving these integrals. It **has** the additional benefit of providing a pedagogical tool for expanding the use of the hyperbolic functions in elementary calculus.

We begin by noting the following:

Theorem 1. $\int \operatorname{sech}(x)\,dx =$ 2 tan–'(e^x)

Proof:

$$\int \operatorname{sech}(x)\,dx=\int\frac{2}{e^x+e^{-x}}\,dx=\int\frac{2e^x}{e^{2x}+1}\,dx=2\tan^{-1}(e^x).$$

Corollary 1. If $\beta = 2\tan^{-1}(e^x)$, then $d\beta=\operatorname{sech}(x)\,dx$.

Corollary 1 indicates that if $\beta = 2\tan^{-1}(e^x)$, then there is a relationship between the hyperbolic functions of x and the trigonometric functions of β. We can now establish expressions for the other hyperbolic functions.

Theorem 2. $\sinh(x) = -$ cot(@).

Proof:

$$\begin{aligned}\sinh(x) &= \sinh\{\ln\left[\tan\left(\frac{\beta}{2}\right)\right]\}\\ &=\frac{1}{2}\left[\tan\left(\frac{\beta}{2}\right)-\cot\left(\frac{\beta}{2}\right)\right]\\ &=\left[\frac{1-\cos(\beta)}{\sin(\beta)}-\frac{\sin(\beta)}{1-\cos(\beta)}\right]\\ &=\frac{[1-\cos(\beta)]^2-\sin^2(\beta)}{2\sin(\beta)[1-\cos(\beta)]}\\ &=\frac{1-2\cos(\beta)+\cos^2(\beta)-1+\cos^2(\beta)}{2\sin(\beta)[1-\cos(\beta)]}\\ &=\frac{-2\cos(\beta)[1-\cos(\beta)]}{2\sin(\beta)[1-\cos(\beta)]}\\ &=-\cot(\beta).\end{aligned}$$

Theorem 3. $\cosh(x)=\csc(\beta)$.

Proof:

$$\begin{aligned}\cosh(x)&=\cosh\left(\ln\left[\tan\left(\frac{\beta}{2}\right)\right]\right)\\ &=\frac{1}{2}\left[\tan\left(\frac{\beta}{2}\right)+\cot\left(\frac{\beta}{2}\right)\right]\\ &=\frac{1}{2}\left[\frac{1-\cos(\beta)}{\sin(\beta)}+\frac{\sin(\beta)}{1-\cos(\beta)}\right]\\ &=\frac{[1-\cos(\beta)]^2+\sin^2(\beta)}{2\sin(\beta)[1-\cos(\beta)]}\\ &=\frac{1-2\cos(\beta)+\cos^2(\beta)+1-\cos^2(\beta)}{2\sin(\beta)[1-\cos(\beta)]}\\ &=\frac{2[1-\cos(\beta)]}{2\sin(\beta)[1-\cos(\beta)]}\\ &=\csc(\beta).\end{aligned}$$

The following corollaries are the direct results of Theorems 2 and 3:

Corollary 2. $\tanh(x) = -\cos(\beta)$.

Proof: $\tanh(x) = \dfrac{\sinh(x)}{\cancel{\cosh(x)}} = \dfrac{-\cot(\beta)}{\cancel{\csc(\beta)}} = -\cos(\beta)$.

Corollary 3. $\operatorname{sech}(x) =$ sin(@).

Proof: $\operatorname{sech}(x) = \dfrac{1}{\cancel{\cosh(x)}} = \dfrac{1}{\cancel{\csc(\beta)}} =$ sin(@).

Corollary 4. $\operatorname{csch}(x) = -\tan(\beta)$.

Proof: $\operatorname{csch}(x)=\dfrac{1}{\sinh(x)}=\dfrac{-1}{\cot(\beta)}=$ - tan(@).

The following examples illustrate how to apply these transformations to some antiderivatives:

Example 1.

$$\begin{aligned}\int\frac{d\beta}{1+\text{sin(@)}-\cos(\beta)}&=\int\frac{\operatorname{sech}(x)\,dx}{1+\operatorname{sech}(x)+\tanh(x)}\\ &=\int\frac{dx}{1+\cosh(x)+\sinh(x)}\\ &=\int\frac{dx}{1+e^x}\\ &=x-\ln|1+e^x|+C \quad \text{[Let } x=\ln(y) \text{ and integrate by partial fractions.]}\\ &=\ln\left|\tan\left(\frac{\beta}{2}\right)\right|-\ln\left|1+\tan\left(\frac{\beta}{2}\right)\right|+C\\ &=\ln\left|\frac{\tan(\frac{\beta}{2})}{1+\tan(\frac{\beta}{2})}\right|+C.\end{aligned}$$

Example 2.

$$\int \frac{d\beta}{3-2\cos(\beta)} = \int \frac{\operatorname{sech}(x)dx}{3+2\tanh(x)}$$
$$= \int \frac{dx}{3\cosh(x)+2\sinh(x)}$$
$$= \int \frac{2e^x\,dx}{5e^{2x}+1}$$
$$= \frac{2}{\sqrt{5}}\int \frac{d(\sqrt{5}e^x)}{5e^{2x}+1}$$
$$= \frac{2}{\sqrt{5}}\tan^{-1}(\sqrt{5}e^x)+C$$
$$= \frac{2}{\sqrt{5}}\tan^{-1}\left[\sqrt{5}\tan\left(\frac{\beta}{2}\right)\right]+C.$$

Example 3.

$$\int \frac{d\beta}{5+4\sin(\beta)} = \int \frac{dx}{5\cosh(x)+4}$$
$$= \int \frac{dx}{\frac{5e^{2x}+5}{2e^x}+4}$$
$$= \int \frac{2e^x dx}{5e^{2x}+8e^x+5}$$
$$= \left(\frac{2}{5}\right)\int \frac{d(e^x)}{(e^x+\frac{4}{5})^2+(\frac{3}{5})^2}$$
$$= \left(\frac{2}{5}\right)\tan^{-1}\left(\frac{4+5e^x}{3}\right)+C$$
$$= \frac{2}{3}\tan^{-1}\left[\frac{4+5\tan(\frac{\beta}{2})}{3}\right]+C.$$

Example 4.

$$\int \sec(\beta)d\beta = -\int \coth(x)\operatorname{sech}(x)dx$$
$$= -\int \frac{\cosh(x)dx}{\sinh(x)\cosh(x)}$$
$$= -\int \frac{dx}{\sinh(x)}$$
$$= -\int \frac{\sinh(x)dx}{\sinh^2(x)}$$

$$= -\int \frac{d[\cosh(x)]}{\cosh^2(x)-1}$$
$$= \coth^{-1}[\cosh(x)]$$
$$= \coth^{-1}[\csc(\beta)]$$
$$= \ln|\sec(\beta)+\tan(\beta)|+C.$$

Example 5.

$$\int \csc(\beta)\,d\beta = \int \cosh(x)\operatorname{sech}(x)\,dx = \int dx = x+C = \ln\left[\tan\left(\frac{\beta}{2}\right)\right]+C.$$

In summary, as an alternative to

$$\sin(\beta) = \frac{2x}{1+x^2},\quad \cos(\beta) = \frac{1-x^2}{1+x^2},\quad \text{and}\quad d\beta = \frac{2dx}{1+x^2},$$

the substitutions

$$\sin(\beta) = \operatorname{sech}(x),\quad \cos(\beta) = -\tanh(x),\quad \text{and}\quad d\beta = \operatorname{sech}(x)\,dx$$

can be used to solve many antiderivatives involving rational functions of $\sin(\beta)$ and $\cos(\beta)$.

Reference

1. S. M. Farrand and N. J. Poxon, Calculus, Harcourt, Brace, Jovanovich, 1984.

Timothy Holland *prepared* this paper while he was teaching at St. Jude High School and enrolled in a master's program at Alabama State University.

CHANGES OF ADDRESS

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THE EASTER DATE PATTERN

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An intriguing date is that of Easter. By a seemingly erratic pattern, it makes **its** appearance on the calendar each year. Sometimes in March and at other times in April. Sometimes very early and at other times, quite late. But always on Sunday. The date of Easter for a given year was **fixed** in A.D. *325* by the Council of Nicaea. In this ancient decree, Easter became accordingly the first Sunday after the first full moon on or **after** the vernal (spring) equinox. As both lunar and solar **cycles** are involved as well as the day of the week pattern, challenging mathematical problems come to light. In particular, how is Easter to be calculated for a given year? What too can be said about the frequency of the various Easter dates and their subtle, hardly noticeable calendar patterns?

Several well known formulas come to mind in pursuit of these and similar mathematical questions. One such approach is a variation on the Easter formula as given by the great mathematician Carl **Friedrich** Gauss **(1777-1855).** Before elaborating however on computational techniques, a brief historical note is in order.

Much diversity characterized the Easter observance pattern prior to the year **325.** Even in later years, recurring problems arose as a consequence of the far-reaching calendar change of *1582.* In that year, the ancient Julian calendar was replaced by the modern calendar of Pope Gregory **XIII** (the Gregorian calendar). The motivation for the change was essentially one of alignment of dates with seasons. Controversy surrounded the new calendar's introduction; various nations were likewise slow in adoption. Although the Gregorian calendar is the one in present worldwide civil use, some today, for ecclesiastical purposes, celebrate Easter in accordance with the ancient Julian calendar. Coincidentally, the Julian and Gregorian calculations of Easter will occasionally give the same date *(as* happened in *1865, 1905,* and *1954* for example).

A **Metonic** cycle **from** ancient times essentially equated *235* full moons with *19* vernal equinoxes. Hence, a time period of *19* years denotes the cycle in which the sun and moon patterns eventually prove commensurable. (The Athenian astronomer **Meton** devised a calendar pattern in *432* B.C. whereby the new moons repeat in *19* year cycles.) More precisely, an integral multiple of one cycle coincided with an integral multiple of the other. However, the calendar reformers of *1582* realized a very slight discrepancy in this equation, namely, the one which blended the lunar cycle with the *19* year solar pattern. The assumption of equality was implicit in the Julian calculation of Easter. The Gregorian correction incorporates the fact that the **Dominical** Letter of a year (the symbol for the year's first Sunday) and the Golden Number (a given year's place in the overall *19* year cycle) will not, in and of themselves, give the exact Easter date.

Because of the complexity of the relationship between the lunar cycle and the solar cycle, various Easter formulas are restricted to but a single century. Each century thus has its own full moon **sequence.** Such a complexity of relationships is accounted for concisely by appropriate references to time called EPACTS, (The word "epact" stems from the Greek and denotes the "age" of the moon in **days** at the start of a new year.) **Mathematicians** can verify (see the Kluepfel reference) that there are exactly *30* epacts as well **as** *30* sets of correspondences involving epacts and Golden Numbers. It is therefore **possible** to construct an Easter formula or set of Easter correspondences which will prove accurate for all time.

It can **also** be shown that the Gregorian calendar's period, namely, **its** perfect date-day cycle of repetition, **is** exactly *400* years. Hence, as December *25, 1994* falls on a Sunday, so **will** Christmas Day 400 years later. **By** examining any *400* year interval of time, it can be establiihed that, for **example,** the thirteenth of a month falls more often on Friday than any other day of the week.

Likewise, it can be proved that Presidential Inauguration Day occurs more often on Sunday than any other day of the week (as last happened in *1985* and will next occur in *2013).*

A more relevant point is that the Easter period can also be calculated. What then is the smallest **interval** of time which implies consistently a perfect cycle of Easter date repetition? **Suppose** a key symbol is associated with each of the thirty Golden Number and Epact associations mentioned above. Let these key **symbols** for convenience be the integers *0* through *29.* It can be shown (see **Kluepfel)** that any cycle of *100* centuries has a new key symbol (number). This accounts for *30 (100)* or **3000** centuries. Yet each of these century intervals is associated with one of the nineteen possible Golden Numbers, no two of which are alike. Accordingly, the Easter period becomes **19(3000)** centuries or *5,700,000* years. Acknowledging **thus** this Easter period of *5,700,000* years, a tabulation of Easter date frequencies becomes possible. Note among other things that *5,700,000* is divisible by *400,* in which case the day of the week pattern (Sunday restriction) is maintained.

VARIATION ON THE GAUSSIAN EASTER FORMULA

The sequence of steps which permits calculating the date of Easter for a particular **year** is given below

1. $\dfrac{\text{year}}{19} = A$ plus remainder B

2. $\dfrac{\text{year}}{100} = C$ plus remainder D

3. $\dfrac{C}{4} = E$ plus remainder F

4. $\dfrac{C+8}{25} = G$ plus discarded remainder

5. $\dfrac{C+1-G}{3} = H$ plus discarded remainder

6. $\dfrac{19B+C+15-(E+H)}{30} =$ quotient (discard) plus remainder Z

7. $\dfrac{D}{4} = K$ plus remainder L

8. $\dfrac{2F+2K+32-(Z+L)}{7} =$ quotient (discard) plus remainder N

9. $\dfrac{B+11Z+22N}{451} = P$ plus discarded remainder

10. $\dfrac{Z+N+114-7P}{31} = Q$ plus remainder R

Then Q denotes the month and R + *1* denotes the day on which Easter falls for a given year, An illustration reinforces the formula. To calculate Easter for the year **1998,** the following letter values are obtained.

1. $A = 105$ $B = 3$
2. $C = 19$ $D = 98$
3. $E = 4$ $F = 3$
4. $G = 1$
5. $H = 6$
6. $Z = 21$
7. $K = 24$ $L = 2$
8. $N = 0$
9. $P = 0$
10. $Q = 4$ $R = 11$

As Easter is given by month Q and day $R+1$, the actual date of Easter for **1998** is April **12.** It is **also** the most common Easter date of the twentieth century (occurring six times).

HOW EARLY AND **HOW** LATE?

Easter may occur as early as March **22.** This last occurred in **1818** and before that in **1761** and **1693.** Such an early occurrence is actually a rarity. Easter will not come *so* early again in this century or **in** the next. Not until the year **2285** will Easter fall on March **22.**

At the other extreme, Easter may come **as** late **as** April **25.** Its last such occurrence was in **1943** and, prior to that, in **1886.** Easter will next occur on this latest possible date in the year **2038.**

All of the above dates relate to the Gregorian calendar. The calculation of Easter dates according to other schemes frequently deviates **from** this as mentioned earlier. For example, Easter Sunday in Russia in **1989** occurred on April **30.** Such a late date s t e m from **Julian** results which are assigned corresponding Gregorian dates.

As noted, there are **35** possible **dates** for Easter Sunday according to the modern Gregorian calendar. Ten such dates are in March; the remaining twenty-five are in April.

THE TWENTIETH CENTURY

The Easter Sunday frequency pattern for the twentieth century appears in the graph **below.** Note that all possible Easter dates are represented except March **22** and April **24.**

EASTER SUNDAY FREQUENCY

the twentieth century **(1901 – 2000)**

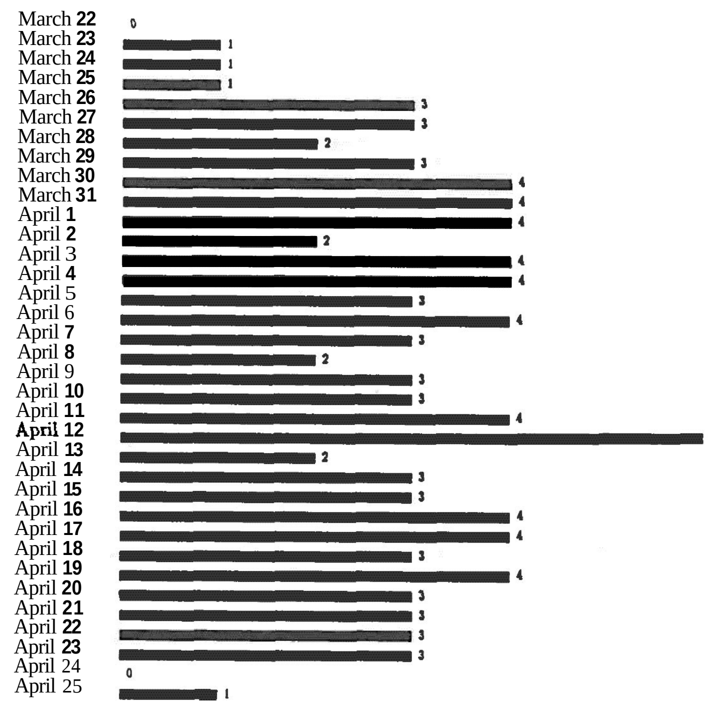

TABLE I

FIRST 2000 YEARS OF THE GREGORIAN CALENDAR

All possible Easter dates appear in this **2000** year period of time. The least frequent date is **March 22,** occurring but **13** times. The most frequent are April 4 and April **10,** each occurring **83** times.

EASTER SUNDAY FREQUENCY

the first two-thousand years of the Gregorian Calendar (1583 – 3582)

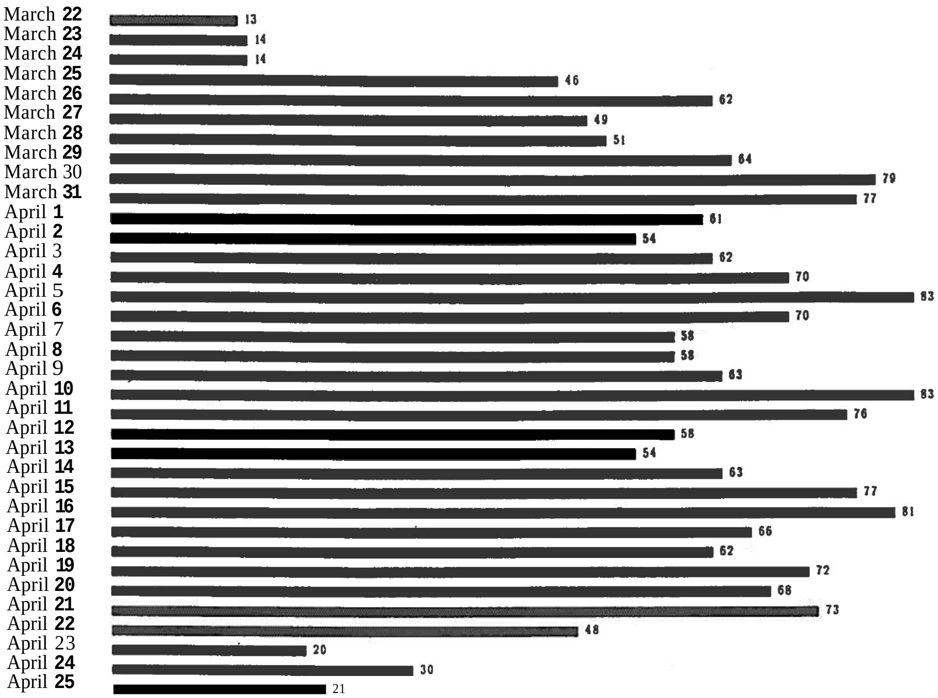

TABLE II

ENTIRE EASTER PERIOD OF 5,700,000 YEARS

Not only do all possible Easter dates appear in this vast period of time (that of a perfect Easter date cycle of repetition), it is also easy to tell which of the dates is the least frequent and which is the most Sequent. Note that March **22** (the least frequent date) occurs **27,550** times. The runner-up is April **25** (occurring **42,000** times). Note likewise that April **19** is the most frequent; it occurs **220,400** times. **This year, 1992,** Easter falls on its most frequent date. The middle Easter date (from March **22** to April **25**) is April **8;** it occurs **192,850** times. Moreover, the average frequency is obtained by dividing **5,700,000** by **35.** This average is approximately **162,857.**

EASTER SUNDAY FREQUENCY

the entire Easter period of **5,700,000** years

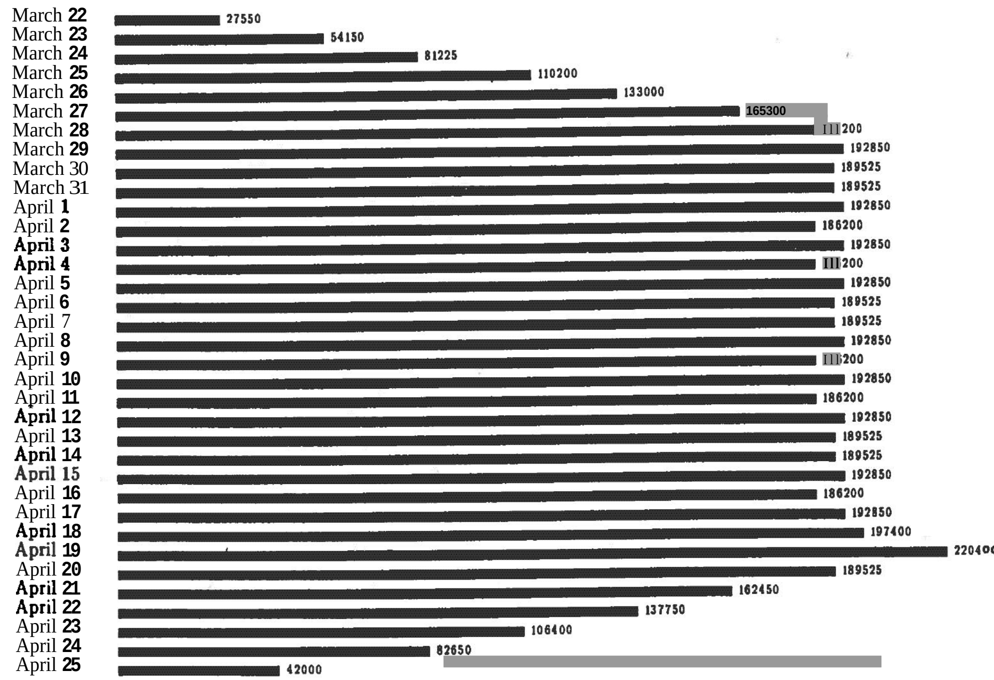

TABLE III

EXPLORATIONS

Various questions and conjectures arise in examining a long list of consecutive Easter dates. Consider for example the thousand year Easter listing given below.

EASTER SUNDAY

far Easter	Vtar Easter	Vtar Easter	Year Easter	Year Easter	Year Easter	Year Easter	Year Easter	Year Easter	Year Easter
1600 Apr 2	1601 Apr 22	1602 Apr 7	1603 Mar 30	1604 Apr 18	1605 Apr 10	1606 Mar 26	1607 Apr 15	1608 Apr 6	1609 Apr 19
1610 Apr 11	1611 Apr 3	1112 Apr 22	1613 Apr 7	1614 tar 10	1115 Apr 19	1616 Air 3	1117 Mar 21	1618 Apr 15	1619 Mar 31
1620 Apr 19	1621 Apr 11	1622 Mar 27	1623 Apr 16	1624 Apr 7	1625 KIT 10	1626 Apr 12	1627 Apr 4	1628 Apr 23	1629 Apr I5
1630 Mar 31	1631 Apr 20	1632 Apr 11	1633 Mar 27	1634 Apr 16	1635 Apr 8	1636 Mar 23	1137 Apr 12	1638 Apr 4	1639 Apr 24
1640 Apr 8	1141 Mar 31	1642 Apr 20	1143 Apr 5	1644 Mar 27	1645 Apr 16	1646 Apr 1	1647 Apr 21	1648 Apr 12	1649 Apr 4
1150 Apr 17	1651 Apr 9	1152 Mar 31	1653 Apr 13	1654 Apr 5	1655 Mar 28	1656 Apr 16	1157 Apr 1	1658 Apr 21	1659 Apr 13
1660 Mar 21	1661 Apr 17	1662 Apr V	IU3 Mar 25	1664 Apr 13	1665 Apr 5	1666 Apr 25	1667 Apr 10	1668 Apr 1	1669 Apr 21
1670 Apr 6	1171 Mar 29	1672 Apr 17	1173 Apr 2	1674 Mar 25	1675 Apr 14	1676 Apr 5	1177 Apr 18	1678 Apr 10	1679 Apr 2
1680 Apr 21	1681 Apr 6	1682 Mar 29	1183 Apr 18	1684 Apr 2	1685 Apr 22	1686 Apr 14	1687 Mar 10	1688 Apr 18	1689 Apr 10
1690 mar 21	1691 Apr I5	1692 Apr 6	1693 Mar 22	IH4 Apr 11	1695 Apr 3	1696 Apr 22	1697 Apr 7	1698 Mar 10	1699 Apr 19
1700 Apr 11	1701 Mar 27	1702 Apr 11	1103 Apr 8	1704 Mar 23	1705 Apr 12	1701 Apr 4	1707 Apr 24	1708 Apr 8	1709 Mar 31
1710 Apr 20	1711 Apr 5	1712 Mar 27	1713 Apr 16	1714 Apr 1	1715 Apr 21	1716 Apr 12	1717 Bar 28	1718 Apr 17	1719 Apr V
1720 Mar 31	1721 Apr 13	1722 Apr 5	1723 mar 28	1724 Apr 11	1721 Apr 1	1726 tor 21	1727 Apr 13	1720 HIT 21	1729 Apr 17
1730 Apr V	1731 Mar 25	1732 Apr 13	1733 Apr 5	1734 Apr 25	1735 Apr 10	1736 Apr 1	1737 Apr 21	1738 Apr 6	1739 mar 29
1740 Apr 17	1741 Apr 2	1742 Mar 25	1743 Apr 14	1744 Apr 5	1745 Apr 18	1746 Apr 10	1747 Apr 2	1748 Apr 14	1749 Apr 6
1750 Mar 29	1751 Apr It	1752 Apr 2	1753 Apr 22	1754 Apr 14	1755 Mar 30	17% Air 18	1757 Apr 10	1738 Mar 26	1759 Air IS
1760 Apr 6	1761 Mar 22	1762 Apr 11	1763 Apr 3	1764 Apr 22	1715 Apr 7	1766 Mar 30	1767 Apr 19	1768 Apr 3	1769 Mar 26
1770 Apr 15	1771 Mar 31	1772 Apr 19	1773 Apr 11	1774 Apr 3	1775 Apr 16	1776 Air 7	1777 Mar 10	1778 Apr 19	1779 Apr 4
1780 Mar 26	1781 Apr 15	1102 Mar 31	1783 Apr 20	1784 Apr 11	1785 mar 27	1786 Apr 11	1787 Apr 8	1788 fit 23	1789 Apr 12
1790 Apr 4	1791 Apr 24	1792 Apr 8	1793 mar 31	1794 Apr 20	1795 Apr 5	1796 Mar 27	1797 Apr 16	1798 Apr 8	1799 Mar 24
1800 Apr 13	1801 Apr 5	1802 Apr 18	1803 Apr 10	1804 Apr 1	1805 Apr 14	1801 Apr 6	1807 ■T 29	1808 Apr 17	1809 Apr 2
1810 Apr 22	1811 Apr 14	1812 Mar 29	1813 Apr 18	1814 Apr 10	1815 mar 21	1811 Apr 14	1817 Apr 1	1818 Mar 22	1819 Apr 11
1820 Apr 2	1821 Apr 22	1822 Apr 7	1823 Mar 10	1824 Apr 18	1825 Apr 3	1826 Mar 21	1827 Apr 15	1828 Apr 6	1821 Apr 19
1830 Apr 11	1811 Apr 3	1832 Apr 22	1833 Apr 7	1834 Mar 30	1835 Apr 19	18% Apr 3	1837 Mar 21	1838 Apr 15	1839 Mar 31
1840 Apr 19	1841 Apr 11	1842 Mar 27	1843 Apr 16	1844 Apr 7	1845 Mar 23	1846 Apr 12	1847 Apr 4	1848 Apr 23	1849 Apr 8
1850 Mar 31	1851 Apr 20	1852 Apr 11	1853 Mar 27	1854 Apr 16	1855 Apr 8	1856 Mar 23	1857 Apr 12	1850 Apr 4	1859 Apr 24
1860 Apr 8	1861 Mar 31	1862 Apr 20	1863 Apr 5	1814 Mar 27	1815 Apr 16	1866 Apr I	1867 Air 21	1868 Apr 12	1869 fir 28
1870 Apr 17	1871 Apr 9	1812 Mar 31	1873 Apr 13	1874 Apr 5	1875 Mar 28	1871 Apr 11	1877 Apr 1	1878 Apr 21	1879 Apr 13
1880 Mar 28	1881 Apr 17	1882 Apr V	1883 far 25	1884 Apr 13	1885 Apr 5	1881 Apr 25	IN7 Apr 10	1888 Apr 1	1889 Apr 21
1890 Apr 1	1811 Mar 29	1812 Apr 17	1893 Apr 2	1894 Mar 25	1895 Apr 14	1896 Apr 5	mi Apr 18	1898 Apr 10	1899 Apr 2
1900 Apr IS	HOI Apr 7	1902 Mar 30	1903 Apr 12	IT04 Apr 3	1905 Apr 23	1906 Apr 15	1907 Mar 31	1908 Apr 19	1909 Apr 11
1910 mar 27	1911 Apr 16	1912 Apr 7	HI3 Mar 23	1914 Apr 12	1915 Apr 4	1916 Apr 23	1917 Apr 8	1918 Mar 31	1919 Apr 20
1920 Apr 4	1921 Mar 27	1922 Apr 11	1923 Apr 1	1924 Apr 20	1925 Apr 12	1926 Apr 4	1927 Apr 17	1928 Apr 8	1929 Mar 31
1930 Apr 20	1931 Apr 5	1932 Mar 27	1933 Apr 11	1934 Apr I	1935 Apr 21	1936 Apr 12	1937 Mar 28	1938 Apr 17	1939 Apr 9
1940 fir 24	1941 Apr 13	1942 Apr 5	1943 Apr 25	1944 Apr 9	1945 Apr I	1946 Apr 21	1947 Apr 6	1948 Mar 28	1949 Apr 17
1950 Apr 9	1951 Mar 25	1952 Apr 13	1953 Apr 5	1954 Apr 18	1955 Apr 10	1956 Apr 1	1957 Apr 21	1958 Apr 1	mi Mar 21
Hid Apr 17	1961 Apr 2	1962 Apr 22	1963 Apr 14	1964 Mar 29	1965 Apr 18	1966 Apr 10	1967 Mar 21	1968 Apr 14	1969 Apr 6
1970 Mar 29	1971 Apr 11	1972 Apr 2	H13 Apr 11	1974 Apr 14	1975 Mar 30	1976 Apr 18	1977 Apr 10	1978 Mar 21	1979 Apr 15
1980 Apr 1	1981 Apr 19	1982 Apr 11	1983 Apr 3	1984 Apr 22	1985 Apr 7	1986 Mar 10	1987 Apr 19	1988 Apr 3	1989 Mar 21
1990 Apr 15	1991 mar 31	1992 Apr 19	1993 Apr 11	1994 Apr 3	1995 Apr 11	1996 Apr 7	1997 Mar 30	1998 Apr 12	1999 far 4
2000 Apr 23	2001 Apr I5	2002 Mar 31	2003 Apr 20	2004 Apr 11	2005 Mar 27	2006 Apr 16	2007 Apr 8	2008 mar 23	2009 Apr 12
2010 Apr 4	2011 Apr 24	2012 Apr 8	2013 Mar 31	2014 Apr 20	2015 Apr 5	2016 Mar 27	2017 Apr 11	2018 Apr 1	2019 Apr 21
2020 Apr 12	2021 Apr 4	2022 Apr 17	2021 Apr V	2024 Mar 31	2025 Apr 20	2026 Apr 5	2027 Mar 28	2028 Apr 11	2029 Apr 1
2030 Apr 21	2031 Apr 13	2032 Mar 28	2033 Apr 17	2034 Apr V	2035 Mar 25	2036 Apr 13	2037 Apr 5	2038 Apr 25	2039 Apr 10
2040 Apr 1	2041 Apr 21	2042 Apr 1	2043 Sir 29	2044 Apr 17	2045 Apr 9	2046 Mar 25	2047 Apr 14	2048 Apr 5	2049 Apr 18
2050 Apr I0	2051 Apr 2	2052 Apr 21	2053 Apr 6	2054 Mar It	2055 Apr 18	20% Apr 2	2057 Apr 22	2058 AT 14	205V Mar 30
2060 Apr 18	2061 Apr 10	2012 Mar 26	2063 Apr I5	2064 Apr 1	2065 Mar 29	2066 Apr I1	2067 Apr 3	2068 Apr 22	2069 Apr I4
2070 Mar 30	2071 Apr 19	2072 Apr 10	2073 Mar 21	2074 Apr 15	2075 Apr 7	2076 Apr 19	2077 Apr 11	2078 Apr 3	2079 Apr 23
2080 Apr 7	2081 far 30	2082 Apr 19	2083 Apr 4	2004 Mar 21	2085 Apr I5	2086 Bar 31	2087 Apr 20	2008 Apr 11	2089 Apr 3
2090 Apr 11	2091 Apr 8	mi Mar 30	2093 Apr 12	2094 Apr 4	2095 Apr 24	2096 Apr 15	2097 mar 31	2098 Apr 20	2099 Air I2

TABLE IV

EASIER SUNDAY

Year Easter	Vtar Easter	Year Easter	Vtir Enter	Year Easter	Year Easter	Year Easter	Year Easter	Year Easter	Year Easter
2100 Mar 28	2101 Apr 17	2102 Apr 9	2103 Mar 25	2104 Apr 13	2105 Apr 5	2106 Apr 18	2107 tor 10	2108 Apr 1	2109 tor 21
2110 Apr 6	2111 mar 21	2112 Apr 17	2113 Apr 2	2114 Apr 22	2115 Apr 14	2116 Mar 21	2117 Apr 11	2111 Apr 10	2119 Mar 21
2120 Apr 14	2121 Apr 6	2122 Mar 29	2123 Apr 11	2124 Apr 2	2121 Apr 22	2126 Apr 14	2127 Mar 10	2121 Apr 18	2129 Apr 10
2130 Mar 26	2131 Apr 15	2132 Apr 6	2133 Apr 19	2134 Apr 11	2131 Apr 3	2136 Apr 22	2137 Apr 7	2138 fir 30	2139 Apr I9
2140 Apr 3	2141 Mar 26	2142 Apr 15	2143 Mar 31	2144 Apr 19	2145 Apr 11	2146 Apr 3	2147 Apr 16	2148 Apr 7	2149 Mar 10
2150 Apr 12	2151 Apr 4	2152 Apr 23	2153 Apr 15	2154 Mar 31	2155 Apr 20	21% Apr 11	2157 Mar 27	2158 Apr 16	2159 Apr 8
2160 Mar 23	2161 Apr 12	2162 Apr 4	2113 Apr 24	2164 Apr 8	2165 Mar 31	2166 Apr 20	2117 Apr 5	2168 Mar 27	2169 Apr 11
2170 Apr 1	2171 Apr 21	2172 Apr 12	2173 Apr 4	2174 Apr 17	2175 Apr 9	2171 Mar 31	2177 Apr 20	2178 Apr 5	2179 mar 21
2180 Apr 16	2181 Apr 1	2182 Apr 21	2183 Apr 13	2184 Mar 28	2185 Apr 17	2186 Apr 9	2187 Mar 25	2188 Apr 13	2189 Apr 5
2190 Apr 25	2191 Apr 10	2192 Apr 1	2193 Apr 21	2194 Apr 1	2195 Mar 29	2196 Apr 17	2197 Apr 9	2198 Mar 25	2199 Apr 14
2200 Apr 1	2201 Apr 19	2202 Apr 11	2203 Apr 3	2204 Apr 22	2205 Apr 7	2206 Mar 30	mi Apr 19	2208 Apr 3	2209 Mar 21
2210 Apr 15	2211 Mar 31	2212 Apr 19	2213 Apr 11	2214 Mar 27	2215 Apr 11	2216 Apr 7	2217 Mar 30	2218 Apr 12	2219 Apr 4
2220 Apr 23	2221 Apr I	2222 Mar 31	2223 Apr 20	2224 Apr 11	2225 Mar 27	2226 Apr 11	2227 Apr 8	2228 Mar 23	2229 Apr 12
2230 Apr 4	2231 Apr 24	2232 Apr 8	2233 Mar 31	2234 Apr 21	2235 Apr 5	2231 mar 27	2237 Apr 16	2238 Apr 1	2239 Apr 21
2240 Apr 12	2211 Apr 4	2242 Apr 17	2243 Apr 9	2244 Mar 31	2245 Apr 13	2241 Apr 5	2247 Mar 28	2248 Apr 16	2249 Apr 1
2250 Apr 21	2251 Apr 13	2252 Mar 28	2253 Apr 17	Z54 Apr 9	2255 Mar 25	2251 Apr 13	2257 Apr 5	2258 Apr 25	2259 Apr 10
2260 Apr I	2261 Apr 21	2262 Apr 6	2263 Mar 29	2264 Apr 17	2265 Apr 2	2266 mar 25	2267 Apr 14	2268 Apr 5	2269 Apr 18
2270 Apr 10	2271 Apr 2	2272 Apr 21	2273 Apr 6	2274 Mar 29	2275 Apr 18	2276 Apr 2	2217 Apr 22	2211 Apr 14	2279 Mar 30
2280 Apr 18	2281 Apr 10	2282 Mar 26	2283 Apr IS	2284 Apr 6	2285 mar 11	2286 Apr 11	2287 Apr 3	2288 Apr 22	2289 Apr 7
2290 Mar 30	2291 Apr 19	2292 Apr 10	2293 mi, 26	2294 Apr 15	2295 Apr 7	2296 Apr 19	mi Apr 11	2298 Apr 3	2299 Apr I6
2300 Apr 8	2301 mar 31	2302 Apr 20	2303 Apr 5	2304 Mar 27	2305 Apr 16	2306 Apr 1	2307 Apr 21	2108 Apr 12	2309 Mar 21
2310 Apr 17	2311 Apr 9	2312 Mar 31	2313 Apr 13	2314 Apr 5	2315 Mar 28	2316 Apr 11	2317 Apr 1	2318 Apr 21	2319 Apr 1
2320 Mar 28	2321 Apr 17	2322 Apr 9	2323 Mar 25	2324 Apr 13	2325 Apr 5	2326 Apr 25	2327 Apr 10	2328 Apr 1	2329 Apr 21
2330 Apr 1	2331 Mar 29	2H2 Apr 17	2133 Apr 2	2334 Mar 25	2335 Apr 14	2336 Apr 5	2Ã^ Apr 18	2338 Apr 10	2339 Mar 26
2340 Apr I4	2341 Apr 6	2342 Mar 29	2343 Apr It	2344 Ã‡p 2	2345 Apr 22	2346 Apr 14	2347 Mar 30	2348 Apr 18	2349 Apr 10
2350 Bar 26	2351 Apr 15	2352 Apr 1	2353 Mar 22	2354 Apr 11	2355 Apr 3	23% Apr 22	2357 Apr 7	2358 mar 30	2359 Apr 19
2360 Apr 3	2361 Mar 26	2362 Apr 15	2363 Mar 31	2364 Apr 19	2365 Apr 11	2366 Apr 3	2367 Apr 16	2318 Apr 7	2169 Mar it
2370 Apr 19	2371 Apr 4	2172 Mar 26	2113 Apr I5	2374 Mar 31	2375 Apr 20	2316 Apr 11	mi Mar 27	2378 Apr I6	2319 Apr 8
:SO Mar 23	2381 Apr 12	2382 Apr 4	2381 Apr 24	2384 Apr 8	2385 Mar 31	2181 Apr 20	2387 Apr 5	2388 mar 27	2389 Apr 16
2390 Apr 8	2391 Mar 24	2392 Apr 12	2393 Apr 4	2394 Apr 17	2395 Apr V	2396 Mar 31	2397 Apr 20	2398 Apr 5	2399 Mar a
2400 Apr 11	2401 Apr I	2402 Apr 21	2403 Apr 13	2404 Mar 28	2405 Apr 17	2406 Apr V	2407 Mar 25	2408 Apr 13	2409 Apr 5
2410 Apr 15	2411 Apr 10	2412 Apr I	2413 Apr 21	2414 Apr 6	2415 Mar 29	2416 Apr 17	2417 Apr 2	2418 Mar 25	2419 Apr 14
2420 Apr 5	1421 Apr 18	2422 Apr 10	2423 Apr 2	2424 Apr 21	2425 Apr 6	2426 Mar 29	2427 Apr 18	2428 Apr 2	2429 Apr 22
2430 Apr 14	2431 Mar 10	2432 Apr 18	2433 Apr 10	2414 Mar 26	2435 Apr 15	2436 Apr 1	2437 Mar 22	2438 Apr 11	2439 Apr 3
2440 Apr 22	2441 Apr 7	2442 Mar 10	2443 Apr 19	2444 Apr 10	2445 Mar 26	2446 Apr 15	2447 Apr 7	2448 Apr 19	2449 Apr 11
2450 Apr 3	2451 Apr I6	2452 Apr 7	2451 Mar 30	2454 Apr 19	2455 Apr 4	24% Mar 26	2457 Apr 15	2458 Mar 31	2459 Apr 20
2460 Apr 11	2461 Mar 27	2412 Apr 16	2463 Apr 8	2464 Mar 30	2465 Apr 12	2466 Apr 4	2467 Apr 24	2468 Apr I5	2419 mar 31
2470 Apr 20	2471 Apr 5	2472 Mar 27	2473 Apr 11	2474 Apr 8	2475 mar 24	2476 Apr 12	2477 Apr 4	2478 Apr 24	2479 Apr 9,
2480 Mar 31	2481 Apr 20	2482 Apr 5	2433 Mar 28	2484 Apr 16	2485 Apr 1	2186 Apr 21	2487 Apr 13	2488 Apr 4	2489 Apr 17
2490 Apr V	2491 Mar 25	2492 Apr 13	2493 Apr 5	2494 Mar 28	2495 Apr 10	2496 Apr 1	2497 Apr 21	2498 Apr 13	2499 mar 29
2500 Apr 18	2501 Apr 10	2502 Mar 26	2503 Apr I5	2504 Apr 1	2505 Mar 22	2506 Apr 11	2507 Apr 3	2508 Apr 22	2509 Apr 7
2510 Mar 30	2511 Apr 19	2512 Apr 3	2513 mar 26	2514 Apr 15	2515 Mar 31	2516 Apr 19	2517 Apr 11	2518 Apr 3	2519 Apr 16
2520 Apr 7	2521 Mar 30	2522 Apr 19	2523 Apr 4	2524 Mar 21	2525 Apr I5	2526 ■T 31	2527 Apr 20	2528 Apr 11	2529 Mar 27
2530 Apr 16	2531 Apr 8	2532 Mar 23	2533 Apr I2	2534 Apr 4	2535 Apr 24	2536 Apr 8	2537 Mar 31	2538 Apr 20	2539 Apr 5
2540 Mar 27	2541 Apr 11	2542 Apr 8	2543 mar 24	2544 Apr 12	2545 Apr 4	2546 Apr 17	2541 Apr V	2548 Mar 31	2549 Apr 20
2550 Apr 5	2551 Mar 28	2552 Apr 11	2553 Apr I	2554 Apr 21	2555 Apr 13	2556 Mar 28	2557 Apr 17	2558 Apr V	2559 Mar 25
2560 Apr 13	2561 Apr 5	2512 mar 28	2563 Apr 10	2564 Apr I	2515 Apr 21	2566 Apr 6	2567 Mar 29	2568 Apr 17	2569 Apr V
2570 Mar 25	2571 Apr 14	2572 Apr 5	2573 Apr 25	2574 Apr 10	2575 Apr 2	2576 Apr 21	2577 Apr 6	2578 Mar 29	2579 Apr 18
2580 Apr 2	2581 mar 25	2582 Apr 14	2583 Mar 10	2184 Apr 18	2585 Apr 10	2586 Mar 26	2187 Apr 15	2588 Apr 1	2189 Mar 21
2590 Apr 11	2591 Apr 3	2592 Apr 22	2593 Apr I4	2594 tor 30	2595 Apr 19	2596 Apr 10	2597 Mar 26	2598 Apr 15	2599 Apr 7

TABLE IV (cont.)

A typical question concerns the month pattern. In particular, "Can Easter occur in March in both of any two consecutive years?" The table above strongly suggests the answer is "no." In order to be certain, one must examine the entire Easter period, or, by consideration of the formula, establish that Q cannot be **3** (the March number) for consecutive year numbers. By a computer analysis of the Q question for the entire Easter period of **5,700,000** years, the answer of "no" is verified. Accordingly, Easter cannot come in March two years in a row. Another question suggested by the long Easter listing is "Can Easter occur on corresponding Sundays in any two consecutive years?" Corresponding Sundays in consecutive years are those Sundays which differ by exactly **52** weeks. For example, April **9, 1950** and April **8, 1951** are corresponding Sundays. The first is an Easter date; the second is not. Knowing for instance that April **12, 1998** is Easter (as calculated earlier), can one now assert for a fact that neither April **13,** Sunday, **1997** nor April **11,** Sunday, **1999** is an Easter date? Once again, by a computer analysis of the Easter period of **5,700,000** years, the answer is "consecutive Easters cannot occur on corresponding Sundays." Other questions likewise stem from the Easter listing. A few are included here for the purpose of additional exploration.

1. It appears that in consecutive years, Easter dates can be no closer **datewise** than **8** days. Consider for example the Easter dates April **18, 1965,** and April **10, 1966.** Note too that the earlier year always seems to contain the later date number. Are these suggested patterns valid?

2. The shortest interval of time separating a given Easter and its next like date occurrence is five years. This happened, for example, on (Easter) March **29, 1959,** and (Easter) March **29, 1964.** What is the greatest interval of time separating a given Easter and its next like date occurrence? Note that long intervals of time are suggested by examining Easter lists. Among them are the Easter dates March **22, 1818,** and March **22, 2285,** which span an interval of **467** years.

3. Is it possible for a decade to consist entirely of April Easters? What is the greatest number of consecutive April Easters possible?

4. **Ponce** de Leon, the European discoverer of Florida, gave the area its name ("Florida" or "flowery Easter") on Easter Sunday in the year **1513.** What was the exact date of this Julian calendar Easter and could it be the same as its projected Gregorian date counterpart, namely, April **6?**

5. Can two like date Easters be exactly **400** years apart? Recall that the period of the Gregorian calendar is **400** years.

The above are but a few of the many questions contained in the mathematical subtlety of the Easter date pattern.

The Gregorian calendar will prove many times out of line with the seasons in the course of **5,700,000** years. Actually, the calendar proves a day in error every **3323** years. In less than **100,000** years, the present calendar's marginal inconsistency with the seasons will magnify and measure roughly a month. Accordingly, it must be stressed that the computations above rest on the assumptions implicit in the Gregorian calendar's construction (a calendar likely to be modified or abandoned in the years ahead). Still, in its present, highly familiar form, it affords an opportunity for the mathematically curious to explore an intriguing pattern of numbers and number relationships.

Appreciation is expressed to Victor Gummersheimer and Johnny Lai for their computer assistance in the preparation of this manuscript.

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ERRATA

The following errors appeared in the Fall, **1991,** issue of the ***Pi Mu Epsilon Journal:***

Page **280** On line **4** and again on line **5,** "$ac - bd \neq 0$" should have read "$ad - bc$ # 0."

Page **292** "Theorem **2"** and "Example **10"** should have been "Theorem 7" and "Example **11."**

'Page **296** Line 7 should have read "365.25/365.25 + 11365.25 = 24 hours/X."

Page **297** On line **5,** "then O_1, O_2, and O_3" should have read "then O_1O_2 and CO_3."

Page **299** On line **4** of Theorem **2,** "$A'_1 = C_2$" should have read "$A'_n = C_2$."

The Editor apologizes for any problems that these errors might have caused.

EXTENDING A FAMILIAR INEQUALITY

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The following problem appeared on the 1973 USA Mathematical Olympiad:

Prove that if $a, b,$ and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3} \tag{1}$$

A simpler version,

$$a^a b^b \geq (ab)^{(a+b)/2},$$

is a familiar exercise in a number of texts.

The usual proofs of (1) use a not particularly simple elementary argument or Jensen's inequality. [See M. S. Klamkin, USA Mathematical Olympiads 1972-1986, MAA, 1988, p. 81.]

We start our proof by noting that if $x > 0$, then

$$x \ln x \geq x - 1 \geq \ln x \tag{2}$$

with equality iff $x = 1$.

We use (2) to extend (1) and then to obtain an important limiting relation for the power mean.

The right side of (2) follows immediately from the observation that $f(x) = x - 1 - \ln x$ has an absolute minimum at $x = 1$ because $f'(x) = 1 - 1/x = 0$ iff $x = 1$, and $f''(x) = 1/x^2$ is positive for $x > 0$. If we now replace x by $1/x$ in $x - 1 \geq \ln x$ we get $x \ln x \geq x - 1$ which completes (2).

Let $A = (a + b + c)/3$ and substitute $x = a/A, x = b/A,$ and $x = c/A$ successively into (2). Adding gives

$$\frac{a}{A}\ln\frac{a}{A} + \frac{b}{B}\ln\frac{b}{A} + \frac{c}{A}\ln\frac{c}{A} \geq \frac{a+b+c}{A} - 3 \geq \ln\frac{a}{A} + \ln\frac{b}{A} + \ln\frac{c}{A}$$

Hence

$$\ln\left[(a/A)^{a/A}(b/A)^{b/A}(c/A)^{c/A}\right] \geq 0 \geq \ln\left(\frac{abc}{A^3}\right)$$

or

$$\left(\frac{a^a b^b c^c}{A^{a+b+c}}\right)^{1/A} \geq 1 \geq \frac{abc}{A^3}. \tag{3}$$

It follows that

$$a^a b^b c^c \geq A^{a+b+c} \geq (abc)^{(a+b+c)/3}. \tag{4}$$

This double inequality gives (1) and somewhat more. Also, there is equality iff $a/A = 1, b/A = 1,$ and $c/A = 1$. That is, iff $a = b = c$.

The power mean, M_r, of order r is defined by

$$M_r = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{1/r},$$

where $a_i > 0$ $(i = 1, 2, \ldots, n)$ and $r \neq 0$ are real numbers. Thus M_1 and M_2 are the arithmetic mean and root mean square. If $n = 3$, then $M_r^r = (a^r + b^r + c^r)/3$.

Putting $x = a^r/M_r^r$, $x = b^r/M_r^r$, and $x = c^r/M_r^r$ in (2) and adding gives

$$\frac{a^r}{M_r^r}\ln\frac{a^r}{M_r^r} + \frac{b^r}{M_r^r}\ln\frac{b^r}{M_r^r} + \frac{c^r}{M_r^r}\ln\frac{c^r}{M_r^r} \geq \frac{a^r + b^r + c^r}{M_r^r} - 3$$
$$\geq \ln\frac{a^r}{M_r^r} + \ln\frac{b^r}{M_r^r} + \ln\frac{c^r}{M_r^r}$$

In a similar way to that used to get (3), it follows that

$$\left[\frac{(a^r)^{a^r}(b^r)^{b^r}(c^r)^{c^r}}{(M_r^r)^{a^r+b^r+c^r}}\right]^{1/M_r^r} \geq 1 \geq \frac{a^r b^r c^r}{M_r^{3r}}.$$

If $r > 0$, raising to the M_r^r/r power gives

$$\frac{a^{a^r} b^{b^r} c^{c^r}}{M_r^{a^r+b^r+c^r}} \geq 1 \geq \left(\frac{abc}{M_r^3}\right)^{M_r^r}$$

or

$$a^{a^r} b^{b^r} c^{c^r} \geq M_r^{a^r+b^r+c^r} \geq (abc)^{(a^r+b^r+c^r)/3} \tag{5}$$

More generally, the same kind of argument can be used to get

$$a_1^{a_1^r} a_2^{a_2^r} \ldots a_n^{a_n^r} \geq M_r^{a_1^r+a_2^r+\cdots+a_n^r} \geq (a_1 a_2 \ldots a_n)^{(a_1^r+a_2^r+\cdots+a_n^r)/n} \tag{6}$$

Inequality (4) is a special case of (5). If $r < 0$, the inequalities in (5) are reversed. Thus, for example, if $r = -1$, then M_{-1} is the harmonic mean and (5) becomes

$$a^{\frac{1}{a}} b^{\frac{1}{b}} c^{\frac{1}{c}} \leq M_{-1}^{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leq (abc)^{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)/3}$$

If $r = 0$, M_r is not defined. However, the geometric mean, $\sqrt[3]{abc}$, is usually denoted by M_0. The standard proof that

$$\lim_{r \to 0} M_r = \sqrt[n]{a_1 a_2 \ldots a_n} \tag{7}$$

uses L'Hospital's Rule and the theory of exponential functions. [See Hardy, Littlewood, and Polya, Inequalities, Cambridge University Press, Cambridge, 1952, p. 15.]

Equation (7) follows at once from the observation that (5) can be written as

$$\left(a^{a^r} b^{b^r} c^{c^r}\right)^{1/(a^r+b^r+c^r)} \geq M_r \geq (abc)^{1/3}.$$

If $r \to 0$, $\left(a^{a^r} b^{b^r} c^{c^r}\right)^{1/(a^r+b^r+c^r)}$ tends to $\sqrt[3]{abc}$ and we get (7) for $n = 3$. The general case can be proved in an analogous manner using (6).

PROOF OF THE CONVERGENCE OF A SEQUENCE OF RADICALS

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The purpose of this paper is to investigate the expression

$$S = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}}\ .$$

We first note that S represents a sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers. Checking a few terms, either by hand or with an easily written computer program, leads us to conjecture that the sequence converges to 3. In order to prove this, we first show that it is a monotone increasing sequence that is bounded above by 3. Thus, $\lim_{k\to\infty} a_k = a$ exists and $a \le 3$. Finally, we show that $a = 3$.

To write S as an increasing sequence and see that it is bounded above by the number 3, we note that

$$a_1 = \sqrt{1+2} \ < \ \sqrt{1+2(4)} \ = \ 3$$

$$a_2 = \sqrt{1+2\sqrt{1+3}} \ < \ \sqrt{1+2\sqrt{1+3(5)}} \ = \ 3$$

$$a_3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4}}} \ < \ \sqrt{1+2\sqrt{1+3\sqrt{1+4(6)}}} \ = \ 3$$

$$\vdots$$

$$a_k = \sqrt{1+2\sqrt{1+3\sqrt{1+\dots+(k-1)\sqrt{1+k\sqrt{1+(k+1)}}}}}$$

$$< \sqrt{1+2\sqrt{1+3\sqrt{1+\dots+(k-1)\sqrt{1+k\sqrt{1+(k+1)(k+3)}}}}} \ = \ 3$$

$$\vdots$$

Since $\{a_k\}_{k=1}^{\infty}$ is an increasing bounded sequence, bounded above by 3, we know that $\lim_{k\to\infty} a_k = a$ exists and $a \le 3$.

Thus we can write

$$a = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}} \qquad (1)$$

To prove that $a = 3$, we construct a sequence $\{b_n\}_{n=1}^{\infty}$ such that $b_n < a \le 3$ for every n, and then show that $\lim_{n\to\infty} b_n = 3$. In order to construct our b_n, we must get some idea of how far each a_k is from 3. If we consider, for example, just the part

$$\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+\dots}}}}\ ,$$

this is clearly at least as big as x, where

$$x = \sqrt{1+6\sqrt{1+6\sqrt{1+6\sqrt{1+\dots}}}}\ .$$

We then note that $\sqrt{1+6x} = x$. By using the quadratic formula on the equation $x^2 - 6x - 1 = 0$, we can see that $x > 6$. We can then compare

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5(6)}}}}$$

to

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5(6+1)}}}}\ ,$$

which is exactly equal to 3. We can now generalize this approach in order to construct the b_n.

Notice that

$$d_n = \sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\dots}}}} \ > \ x_n$$

where

$$x_n = \sqrt{1+(n+1)\sqrt{1+(n+1)\sqrt{1+(n+1)\sqrt{1+\dots}}}} \ = \ \sqrt{1+(n+1)x_n}\ .$$

Since $x_n^2 - (n+1)x_n - 1 = 0$, the quadratic formula shows that $x_n > n+1$.

By replacing d_n by $n+1$ in (1), we get b_n, where

$$b_n = \sqrt{1+2\sqrt{1+3\sqrt{1+\dots+(n-1)\sqrt{1+n(n+1)}}}}\ .$$

Since $n+1 < x_n < d_n$, we have $b_n < a$. Thus, for every n, $b_n < a \le 3$.

To complete our proof, we need only show that $\lim_{n\to\infty} b_n = 3$. To do this, we first note that if $0 < x < y$, $0 < w$, and $0 < u < 1$, then $u < \sqrt{u}$, and

$$\frac{x}{y} < \frac{1+wx}{1+wy} < \frac{\sqrt{1+wx}}{\sqrt{1+wy}}.$$

It follows that

$$\frac{n+1}{n+2} < \frac{\sqrt{1+n(n+1)}}{\sqrt{1+n(n+2)}} < \frac{\sqrt{1+(n-1)\sqrt{1+n(n+1)}}}{\sqrt{1+(n-1)\sqrt{1+n(n+2)}}} < \dots < \frac{b_n}{3} < \frac{a}{3} \le \frac{3}{3} = 1,$$

where

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+\dots+(n-1)\sqrt{1+n(n+2)}}}}\ .$$

Since $\lim_{n\to\infty} \dfrac{n+1}{n+2} = 1$, it follows that $\lim_{n\to\infty} b_n = 3$. Thus $a = 3$, and our proof is complete.

A CLOSED FORM FOR A FAMILY OF SUMMATIONS

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Let p be an integer such that $p \geq 2$. It can be shown, by using the asymptotic relationship

$$\binom{n}{p} \sim \frac{n^p}{p!} \quad \text{as } n \to \infty$$

from page 33 of [2], that

$$I(p) = \sum_{n=p}^{\infty} \binom{n}{p}^{-1} \tag{1}$$

converges. The series (1) was evaluated in [1] by using partial sums. In this paper, a closed form for (1) will be obtained by using special functions. The special functions that will be used are reviewed first.

The gamma function is denoted by $\Gamma(x)$ and defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt$$

for $x > 0$. The gamma function has the property that $\Gamma(x+1) = x\Gamma(x)$ provided x is neither zero nor a negative integer. In particular, for $n = 0, 1, 2, \ldots,\ \ \Gamma(n+1) = n!$.

The factorial function is defined by

$$(a)_n = a(a+1)\cdots(a+n-1) \text{ for } n \geq 1 \text{ and } (a)_0 = 1 \text{ for } a \neq 0.$$

In particular, $n! = (1)_n$ and, from page 9 of [2], $(a)_{n+k} = (a+n)_k(a)_n$.

The (Gaussian) hypergeometric function is denoted by ${}_2F_1(a,b;c;x)$ and defined by

$${}_2F_1(a,b;c;x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!}$$

provided c is neither zero nor a negative integer. If none of the parameters a, b, or c are zero or a negative integer, it is known that this series is absolutely convergent for $|x| < 1$, divergent for $|x| > 1$, and is absolutely convergent for $|x| = 1$ provided $a + b - c < 0$.

To evaluate (1), first notice that $I(p)$ can be written as

$$I(p) = p! \sum_{n=p}^{\infty} \frac{(n-p)!}{n!},$$

$$= p! \sum_{n=0}^{\infty} \frac{n!}{(n+p)!}. \tag{2}$$

However,

$$\frac{n!}{(n+p)!} = \frac{(1)_n}{(1)_{p+n}},$$

$$= \frac{(1)_n}{(1+p)_n (1)_p}. \tag{3}$$

Substituting (3) into (2) and simplifying yields

$$I(p) = \sum_{n=0}^{\infty} \frac{(1)_n}{(1+p)_n}.$$

Hence,

$$I(p) = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(1+p)_n\, n!}$$

$$= {}_2F_1(1,1;1+p;1). \tag{4}$$

It has been shown in [3], page 49, that if $\mathrm{Re}(c-a-b) > 0$ and if c is neither zero nor a negative integer,

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Therefore, identity (4) becomes

$$I(p) = \frac{\Gamma(p+1)\Gamma(p-1)}{\Gamma(p)\Gamma(p)}. \tag{5}$$

Since $\Gamma(x) = (x-1)\Gamma(x-1)$, (5) simplifies to give

$$\sum_{n=p}^{\infty} \binom{n}{p}^{-1} = \frac{p}{p-1}.$$

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INQUIRIES

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858, 919-757-6414.

WHAT IS "LOCALLY COMPACT"?

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Each textbook in Topology has its own way of defining what it means for a space to be locally compact. Some authors make an effort to give an equivalent characterization under some additional assumptions about the topological space (see [2]). Essentially there are four wncepts with the name of local compactness and the relations among these have been only partially studied. Even though local compactness, by its very title, is a local property (recall that a property is said to be local if it can be specified for any single point in the space), there has been only global study (that is, a study of the spaces where the property is assumed for every point in the space) of it in the literature. In this paper, we study it locally at a point. Implications among these concepts will be discussed at a particular point. Moreover, we present examples to help understand the impossibility of reverse implications.

Throughout this paper X represents an arbitrary topological space and x denotes a fixed point of X. Schnare [3] discussed two definitions of local compactness, which are rephrased here to define them as properties of space X at a point x as follows: A topological space X is called weakly locally compact, or simply w-compact, at x iff there is a compact neighborhood of x in the space X. X is called mildly locally compact, or m-compact, at x iff there is a neighborhood of x whose closure is compact. A topological space is said to be 1-compact iff it is 1-compact at each of its points where "1" is "w", "m", or any other letter that makes sense in the following discussion. Schnare [3] showed that a w-compact space is m-compact iff the closure of any compact set is compact. Later, Gross [4] introduced a third definition of local compactness, which is modified here as a property at a particular point x. A space X is called bit locally compact, or b-compact, at x iff each neighborhood of x contains a compact neighborhood of x.

It is well known that all these wncepts are equivalent in Hausdorff spaces and regular spaces. In fact, in such spaces, these are equivalent to one more concept called strongly locally compact. X is said to be strongly locally compact, or s-compact, at x iff each neighborhood of x contains a compact closed neighborhood of z. The particular choice of terminology becomes apparent after observing that s-compact is strongest, w-compact is the weakest, and b-compact, m-compact lie in between for any general spaces. That is, we have the following implications in any general topological space X at the point x:

$$\begin{tikzcd} & \text{b-compact} \arrow[dr] & \\ \text{s-compact} \arrow[ur] \arrow[dr] & & \text{w-compact} \\ & \text{m-compact} \arrow[ur] & \end{tikzcd}$$

These implications are strict. Moreover, b-compact and m-compact are incomparable in a general topological space.

Even though compact spaces are obviously m-compact (and thus w-compact), compactness does not imply either s-compactness or b-compactness. Consider the one-point compactification of the space Q of rational numbers. This is a $T_{1\frac{1}{2}}$-space (a space in which each compact set is closed). It

can be shown that it is neither s-compact nor b-compact. This example also tells us that even in compact $T_{1\frac{1}{2}}$-spaces

$$\text{m-compact} \not\rightarrow \text{b-compact}$$

at x. However, b-compact certainly implies m-compact in $T_{1\frac{1}{2}}$-spaces. In fact, this implication holds even under a weaker assumption on the topological space. To explain this assumption, we need the following definition. A space is called an **R-space** iff the closure of a compact set is compact. Clearly any regular or $T_{1\frac{1}{2}}$- (hence T_2-) space is an R-space. since m-compactness at x is equivalent' to the statement that there is a compact closed neighborhood of x in X, it is immediate that in any R-space

$$\text{b-compact} \longrightarrow \text{m-compact}$$

at x and

$$\text{m-compact} \longmapsto \text{w-compact}$$

at x. Of course, b-compact does not imply m-compact in general spaces. Gross [4] has an example of a b-compact normal space which is not m-compact. An easy example is the following: Consider an infinite set X with a distinguished point x in which a set is declared to be open if it is either empty or it contains z. This is a T_0-space (a space in which distinct points have distinct closures) which is b-compact at x but not m-compact.

An infinite set with cofinite topology reveals that even in compact, T_1 - and R-spaces

$$\text{b- and m-compact} \not\rightarrow \text{s-compact at } x.$$

But in-compact and s-compact at x are equivalent in a topological space which is T_2 at x. A space X is called T_2 at x iff for any point y of X different from x, there exist two disjoint open sets G and H in X containing y and x, respectively. It is easy to verify that a topological space X is T_2 at a point x iff to each compact set A not containing x there correspond two disjoint open sets L and M such that $A \subseteq L$ and x 6 M.

Let us show that if X is T_2 at x and m-compact at x then it is s-compact at x. Let G by any open set containing z. Let N be a compact closed neighborhood of x (by m-compactness at x, N exists). Write $A = N \cap G'$, where G' represents the complement of G. Clearly A is a closed subset of the compact space N and hence compact. Since $x \notin A$ and X is T_2 at x, there are disjoint open sets L and M such that $A \subseteq L$ and $x \in M$. Now $M \subseteq L'$ and

$$\overline{M} \subseteq L' \subseteq A' = N' \cup G$$

(the bar indicates the closure of the set), which means $\overline{M} \cap N \subseteq G$. Thus $H = M \cap N^{o}$ (N^{o} is the interior of N) is an open set containing x and

$$\overline{H} \subseteq \overline{M} \cap \overline{N} = \overline{M} \cap N \subseteq G.$$

Moreover, $\overline{H}$, being a closed subset of compact set N, is compact. This shows that X is s-compact at z.

At this point note that, for any R-space that is T_2 at x the implications

$$\text{w-compact} \longrightarrow \text{m-compact} \longrightarrow \text{s-compact}$$

hold at x, hence all compactness concepts are equivalent.

Notice that a space which is s-compact at x is **regular** at x; **that** is, to each open set G containing x there corresponds an open set H such that $x \in H \subseteq \overline{H} \subseteq G$. In fact, this property of the space assures the equivalence of all these concepts. To prove this, let us assume that X is w-compact at x and regular at x. We show that X is s-compact at x. Let G be any open set containing x. Since X is w-compact at x, there is a compact neighborhood N of x. Put A = $G \cap N^0$. Then by regularity at x, there exists an open set H such that

$$x \in H \subseteq \overline{H} \subseteq A.$$

Clearly $\overline{H}$ is compact (because a closed subset of a compact space is compact) and $\overline{H} \subseteq$ G. Thus X is **s-compact** at x. Thus all these concepts are equivalent in spaces which are regular at x, T_2 at x with **R-property,** or Hausdorff spaces.

We close our discussion with an analysis of some of the standard properties of local compact spaces. Clearly any local compactness is x is closed hereditary **(i.e.,** preserved under closed subspaces). However, only **s-** and b-compactness are open hereditary **(i.e.,** preserved under open subspaces). The one point **compactification** of the space Q is m-compact (hence w-compact) in which the open set Q is neither m-compact nor w-compact. A w-compact dense subset B of a $T_{1\frac{1}{2}}$**-space** X is open. Indeed, suppose **b** $\in B$. Since **B** is w-compact, there exist a compact subset C of B and an open subset G of X such that $b \in B \cap G \subseteq$ C. C is closed in X, because X is a $T_{1\frac{1}{2}}$**-space.** Since B is dense in X and G is open in X, $\overline{G} = \overline{B \cap G}$. Thus

$$b \in G \subseteq \overline{G} = \overline{B \cap G} \subseteq \overline{C} = C \subseteq B.$$

This shows that B is a neighborhood of **b.** This being true for any $b \in$ B, B is open.

References

1. **J.** L. Kelly, ***General Topology,*** D. Van **Nostrand** Company, Inc., Princeton, New Jersey, (1966).
2. A. Wilansky, ***Topology for Analysis,*** Ginn and Company, Massachusetts, (1970).
3. P. S. Schnare, "Two Definitions of Local Compactness," ***American Mathematical Monthly 72*** **(1965),** pp 764–765.
4. J. L. Gross, "A Third Definition of Local Compactness," ***American Mathematical Monthly 74*** **(1976),** pp 1120–1122.

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert M. Woodside, **Secretary-**Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858 and to Richard L. **Poss,** Editor, St. Norbert College, De Pere, WI 54115.

ON THE CALCULUS OF RESIDUES

Prem N. Bajaj
The Wichita State University

In this note we give a paradox in the calculation of residues at a pole; a paradox in the sense that an incorrect procedure gives a correct answer. Some of the well-known examples of this type are: the incorrect cancellation of 6 in 16/64, of 9 in 19/95, or of 2 in $(1+x)^2/(1-x^2)$ gives the correct answer. See also **[1].**

Let $f(z) = g(z)/z^n$ where g is analytic and has a zero or order m at the origin; m, n being positive integers and m $< n$. At $z = 0$, f has a pole of order n − m and we discuss its residue R.

Considering, INCORRECTLY, <u>f to be a pole of order n</u>, at the origin, we have,

$$R = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left(z^n\frac{g(z)}{z^n}\right)\Big|_{z=0} = \frac{1}{(n-1)!}g^{(n-1)}(0). \tag{1}$$

However, the pole of f at $z = 0$ is actually of order n − m, and, so,

$$R = \frac{1}{(n-m-1)!}\frac{d^{n-m-1}}{dz^{n-m-1}}\left(z^{n-m}\frac{g(z)}{z^n}\right)\Big|_{z=0} = \frac{1}{(n-m-1)!}\frac{d^{n-m-1}}{dz^{n-m-1}}\left(\frac{g(z)}{z^m}\right)\Big|_{z=0} \tag{2}$$

Now let $g(z) = z^m G(z)$ so that G is analytic at $z = 0$ and G(0) $\neq$ 0. Using Leibnitz's theorem, we have

$$g^{(n-1)}(z) = z^m G^{(n-1)}(z) + \binom{n-1}{1} m z^{m-1} G^{(n-2)}(z) + \ldots + \binom{n-1}{m} m! G^{(n-1-m)}(z)$$

so that

$$g^{(n-1)}(0) = \frac{(n-1)!}{(n-m-1)!}G^{(n-1-m)}(0),$$

reducing (2) to (1).

Incorrectly obtained result (1) can also be seen to be true by using the power series

$$\sum_{k=0}^{\infty}\frac{z^k}{k!}g^k(0)$$

of g. However, the above approach illustrates an application of Leibnitz's theorem – generally forgotten or ignored by students – for finding the nth derivative of the product of two functions.

Reference

1. R. **Katz** and S. Venit, "Partial Differentiation of Functions of a Single Variable," Pi Mu ***Epsilon Journal 7*** **(1982),** 405–406.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME 04469. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) property identified with name and address. Solutions to problems in this issue should be mailed by December 15, 1992.

Correction

761. [Fall 1991] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine all functions $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \frac{1}{f(x)} - \sum_{n=0}^{\infty} (-1)^n a_n x^n.$$

The error was that the exponent on the (-1) was incorrectly given as $n + 1$.

Problems for Solution

771. *Proposed by Alan Wayne, Holiday, Florida.*

In the base six addition

$$EVE + EVE + EVE + AND = 1310$$

the digits of the addends have been unambiguously replaced by letters. Restore the digits. Where was EVE?

772. *Proposed by Robert C. Gebhwdt, Hopatcong, New Jersey.*

Let $xx44$ be a four-digit number and yy be a two-digit number in base $b > 4$. Find x and y in terms of b so that $(yy)^2 = xx44$ in every such base $b > 4$ (such as $88^2 = 7744$ in base ten).

773. *Proposed by Leon Bankoff, Los Angeles, California*

In a given circle (O) a chord CD is drawn to intersect diameter AOB at point E. Three circles are inscribed, the first two in the sectors BEC and BED, and the third in the opposite segment CED. Let the circle in sector BEC touch CE at J and let the circle in sector BED touch DE at N. See the figure. If the three inscribed circles have equal radii,

a) show that CD is perpendicular to AS,

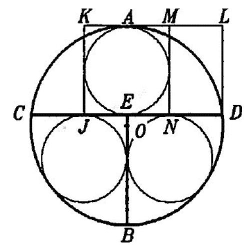
b) find the ratio AE/EB,

c) find the ratio AD/AB,

d) find the ratio CD/AB,

e) show that the rectangle $JKMN$ on JN as base and with opposite side KM passing through A circumscribes the third inscribed circle, and

f) show that the rectangles $JKLD$ and NMW are golden rectangles.



Problem 773

774. *Proposed by Robert C. Gebhwdt, Hopatcong, New Jersey.*

The first player in a game who acquires 250 points is the winner. Because player A is a better player than player B, he gives player B a 50-point handicap. Similarly player B gives player C a 50-point handicap and player C gives player D a 50-point handicap. What handicap should player A give player D?

775. *Proposed by Norman Schaumberger, Bronx Community College, Bronx, New York.*

If H is the harmonic mean of the positive numbers $a_1, a_2, \ldots, a_n$, prove that

$$H^{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \geq a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_n^{\frac{1}{a_n}}.$$

776. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Let n be a fixed positive integer and let

$$P_k = 1^k + 2^k + \cdots + n^k.$$

Write as a polynomial in P_1 the expression

$$15^4(P_1^4 + P_2^4 + P_3^4 + P_4^4).$$

777. *Proposed by Seung-Jin Bang, Seoul, Korea*

It is well known that $\ln(n + 1) < S_n < \ln n$, where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

It is also known (*Out Mathematicorum* 11 (1985) p. 109) that

$$n(n+1)^{1/n} - n < S_n < n - (n-1)n^{-1/(n-1)}.$$

Prove that

$$\ln(n+1) < n(n+1)^{1/n} - n \quad \text{and} \quad n - (n-1)n^{-1/(n-1)} < 1 + \ln n$$

for all $n \geq 2$.

778. ***Proposed** by Laura L. Kelleher and Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

It is readily established that the arc length along the curve $y = \cosh x$ on any interval $[a,b]$ and the area under the graph of this same function on this same interval are numerically equal. For what other functions, if any, is this curious fact true?

779. ***Proposed** by W. Moser, McGill University, Montreal, Canada,*

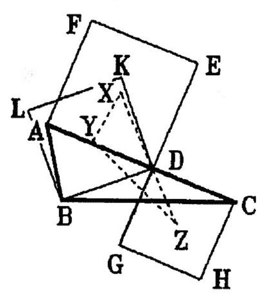
If $0 < a \leq x \leq y \leq 1/a$, then prove that

$$x + \frac{1}{x} \leq a + \frac{1}{a}, \quad \frac{x}{y} + \frac{y}{x} \leq \frac{y}{a} + \frac{a}{y},$$

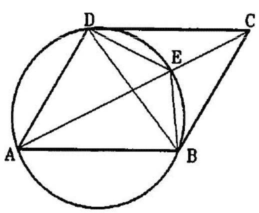
$$\frac{x}{y} + \frac{y}{x} \leq ax + \frac{1}{ax}, \quad \text{and} \quad (x+y)\left(\frac{1}{x} + \frac{1}{y}\right) \leq \left(a + \frac{1}{a}\right)^2.$$

780. ***Proposed** by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Let $ABCD$ be a parallelogram with $\angle A = 60°$. Let the circle through A, B, and D intersect AC at E. See the figure. Prove that $BD^2 + AB \cdot AD = AE \cdot EC$.



Problem 781



Problem 780

781. ***Proposed** by the late Jack Garfunkel, Flushing, New York.*

Erect squares $ADEF$, $BDKL$, and $CDGH$ as shown in the figure, on the segments AD, DC, and BD, where D is any point on side CA of given triangle ABC. Let X, Y, and Z be the centers of the erected squares. Prove that triangles ABC and XYZ are similar and the ratio of similarity is $\sqrt{2}$.

782. ***Proposed** by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

In O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Gronigen, 1969, item 12.55, p. 118, it is stated that for a triangle ABC with no angle $\geq 2\pi/3$ that

$$2(R_1 + R_2 + R_3)^2 \geq (a^2 + b^2 + c^2) + 4F\sqrt{3},$$

where R_1, R_2, and R_3 are the respective distances from an arbitrary point P inside the triangle to its vertices, a, b, and c are the triangle's side lengths, and F is its area. Item 12.55 further states that for a triangle in which $\angle A \geq 2\pi/3$,

$$(R_1 + R_2 + R_3)^2 \geq (b + c)^2.$$

Show that the first inequality is true for all triangles.

783. ***Proposed** by the late Jack Garfunkel, Flushing, New York.*

If, A, B, and C are the angles of a triangle ABC, then prove that

$$\frac{\sum \sin^2 A}{\sum \cos^2\left(\frac{A}{2}\right)} \geq \frac{\prod \sin A}{\prod \cos\left(\frac{A}{2}\right)}.$$

Solutions

403. **[Fall** 1977, **Fall** 1983, Fall **1984]**. ***Proposed** by David L. Silverman, West Los Angeles, California*

Two players play a game of "Take It or Leave It" on the unit interval (0,1). Each player privately generates a random number from the uniform distribution and either keeps it as his score or rejects it and generates a second number which becomes his score. Neither player knows, prior to his own play, what his opponent's score is or whether it is the result of an acceptance or a rejection. (However, variants based on modifying this condition, either unilaterally or bilaterally, are interesting.)

The scores are compared and the player with the higher score wins $1.00 from the other.

a. What strategy will give a player the highest expected score?

b. What strategy will give a player the best chance of winning?

c. If one player knows that his opponent is playing so as to maximize his score, how much of an advantage will he have if he employs the best counter-strategy?

II. ***Comment and solution** by Peter Griffin, California State University Sacramento, California*

Unfortunately the published solution [Fall 1984] is incorrect in two major particulars. Part (b) is coincidentally correct because the expectation for $m = (\sqrt{5} - 1)/2$ turns out to be itself. This is not a valid argument for establishing that this strategy will beat any other strategy, however. The answer to part (c) is wrong because it is based on the same fallacious reasoning as (b).

Both parts (b) and (c) have nothing to do with expectation, but involve how often one strategy will do better than another. Here is a true historical analogy: In the 1960 World Series the New York Yankees averaged 8 runs to only 4 runs for the Pittsburgh Pirates, yet they lost the World Series. The Pirates won more often (4 games to 3), but when the Yankees won, they tended to win by many, many more runs. I will sketch briefly how to solve parts (b) and (c).

"Using an a-rule" shall mean keeping your first random number if it exceeds the criterion number a, otherwise discarding it and being left with the second random number regardless of what it is. The distribution rule of one's final score using an a-rule is $F_a(x) = \Pr[A \le x]$. This function has two formulas:

for $x \le a$, $F_a(x) = \Pr[R_1 \le a]\cdot\Pr[R_2 \le a] = or$, while

for $a < x$, $F_a(x) = \Pr[a < R_1 \le x] + \Pr[R_1 \le a]\cdot\Pr[R_2 \le a]$

$$= (x - a) + ax = (a + 1)x - a.$$

From these formulas we derive the density function for an a-rule to be $f_a(x) = F_a'(x)$, so

$f_a(x) = a$ if $x \le a$, and $f_a(x) = a + 1$ if $x > a$.

The density is integrated over various intervals to End the probability of being in the interval. To find the probability that the random score A is approximately equal to x, we use $f_a(x)\,dx$.

The probability that an a-rule beats a b-rule, assuming $a \le b$, is found by approximating $\Sigma \Pr[A = x]\cdot\Pr[B < x]$ with $\Sigma F_b(x)\cdot f_a(x)\,dx$, which gives $\Pr[A > B] = \Sigma F_b(x) f_a(x)\,\Delta x$, and hence

$$\int_0^1 F_b(x)\ f_a(x)\ dx$$

$$= \int_0^a bx\cdot a\ dx + \int_a^b bx\cdot(a + 1)\ dx + \int_b^1 [(b + 1)x - b]\,(a + 1)\ dx$$

$$= \frac{1}{2} + \frac{(a - b)(1 - ab - b)}{2} = P(a, b).$$

Note that $P(a,b)$ is a quadratic in each variable separately. First, fix b and maximize $P(a,b)$ as a function of a, yielding

$$\frac{\partial P}{\partial a} = \frac{1 - b - 2ab + b^2}{2} = 0 \text{ implies } \hat{a} = \frac{b^2 - b + 1}{2b}.$$

For $b = 1$, this gives 6 = 1/2. But for $b = 1/2$, $\hat{a} = 3/4$, which is not admissible, being greater than 1/2 Admiisibiility of 6 requires that $(b^2 + b + 1)/2b \le b$, which implies that $b^2 + b - 1$ a 0, and hence that $b \ge m = (\sqrt{5} - 1)/2$.

Next fix a and maximize $P(a,b)$, the probability that B loses to A. This requires that

$$\frac{\partial P}{\partial b} = \frac{-a^2 - a - 1 + 2ab + 2b}{2} = 0 \text{ implies } \hat{b} = \frac{a^2 + a + 1}{2(a + 1)}.$$

Admissibility of $\hat{b}$ follows from $\hat{b} \ge a$, implying $(a^2 + a + 1)/(2(a + 1)) \ge a$, so $a^2 + a - 1 \le 0$, and finally $a \le m = (\sqrt{5} - 1)/2$. For $a = 1/2$, we get $\hat{b} = 7/12$ and $P(1/2, 7/12) = 95/192$, so the 7112-rule beats the 1/2-rule $1 - 95/192 = 97/19$ of the time. This is better than

$$1 - P\left(\frac{1}{2}, \frac{5}{8}\right) = 1 - \left(\frac{1}{2} + \frac{-\frac{1}{8}\left(1 - \frac{5}{16} - \frac{5}{8}\right)}{2}\right) = \frac{1}{2} + \frac{1}{256}.$$

It is no surprise that the published solution mentioned simulations that gave 50.4% wins, since $1/2 + 1/256 = 0.504$. The 7112-rule gives $1/2 + 1/19 = 0.505$. So, 7/12 is the answer to Part (c), not 5/8.

Because m satisfies the relation $m = 1/(1 + m)$ and $1 = m + m^2$, it is not hard to show that $P(m,b) > 1/2$ for all $b > m$ and $P(b,m) < 1/2$ for all $b < m$, which demonstrates that m will beat any rule other than itself, which it ties. It provides a saddle point of the function $P(a,b)$, namely (m,m).

The following BASIC program simulates the 7/12 strategy versus the 1/2 strategy 10,000,000 times, printing the results every 10,000 games. To simulate the 5/8 strategy, replace each 7 by 5 and each 12 by 8 in line 12. My results of running this program were that 7/12 beats 1/2 with frequency 0.505215 and 5/8 beats 1/2 with frequency 0.503897, which compare with the ideals 97/19 = 0.505218 and 129/256 = 0.503906.

```
2 FOR J = 1 TO 1000
3 RANDOMIZE
4 FOR I = 1 TO 10000
6 A = RND
8 IF A < .5 THEN A = 2*A
10 B = RND
12 IF B < 7/12 THEN B = 12*B/7
14 IF A > B THEN 18
16 W = W + 1
18 NEXT I
20 PRINT J, 10000*W/J
22 NEXT J
```

731. [Spring 1990, Spring 1991] Proposed by Roger *Pinkham*, Stevens Institute of Technology, *Hoboken*, New Jersey.

a) Show that on the lattice points in the plane having integer coordinates one cannot have the vertices of an equilateral triangle.

*b) What about a tetrahedron in 3-space?

VI. Comment by *Seung-Jin* Bang Seoul, Korea.

In The Newsletter of the Korean Mathematical Society, No. 27 (July 1991) p. 17, there appears a solution by a colleague and me to the generalization of part (a) that states that no regular $(2n + 1)$-gon can have all rational coordinates in the Euclidean plane. We furthermore point out that the result is true for any regular n-gon provided that n is not a power of 2.

733. [Fall 1990, Fall 1991] Proposed by Roger *Pinkham*, Stevens Institute of Technology, *Hoboken*, New Jersey,

If $p(x)$ is a polynomial and $p(x) \ge 0$ for all x, then

$$p + p' + p'' + \dots \ge 0$$

for all x.

IV. *Solution by David Yavenditti, Alma, Michigan.*

Let $S(x) = p(x) + p'(x) + p''(x) + \cdots$, so that $S'(x) = p'(x) + p''(x) + p'''(x) + \cdots$. Then $S(x) = p(x) + S'(x)$. Sicep is a polynomial that is always nonnegative, thenp attains a minimum. Now S is a polynomial with the same leading term as p, so S also must have a minimum. Since S is a polynomial, then S attains its minimum value at $x = c$ only if $S'(c) = 0$. Then, for all real x,

$$S(x) \geq \min\{S(x)\} = S(c) = p(c) + S'(c) = p(c) \geq 0.$$

745. [Spring 1991] *Proposed by Alan Wayne, Holiday, Florida.*

Find *all* solutions to

$$\begin{array}{r} ENID \\ +\ \ DID \\ \hline DINE. \end{array}$$

I. *Solution by Victor G. Feser, University of Mary, Bismarck, North Dakota.*

Since there are four symbols, we solve the problem in base B, where $B \geq 4$. From the B^3 column, $D = E + 1$. From the units column $W \to E$, where the arrow is read "yields" and is equivalent to "congruent mod B." It signifies that 1 may be carried into the next column. Then $2E + 2 \to E$, so $E + 2 \to 0$, whence $E = B - 2$. Now $D = B - 1$ and 1 is carried to the B column (the tens column in base ten).

If 1 is carried to the B^2 column, then it becomes $1 + N + (B - 1) \to I$, and we have $N = I$, which is not allowed. So the B and B^2 columns must read $21 + 1 = N$ and $N + (B - 1) = I + B$. Thus $N = I + 1$ and, from the B column, $I = 0$ and $N = 1$. Hence, for each base $B \geq 4$, the unique solution is

$$\begin{array}{cccc} (B-2) & 1 & 0 & (B-1) \\ & (B-1) & 0 & (B-1) \\ \hline (B-1) & 0 & 1 & (B-2). \end{array}$$

To complete the problem we show that there is no solution for any negative base $B \leq -4$. Since successive powers of a negative number alternate signs, if we carry 1 from a column in an addition, it carries into the next column as -1. From the B^3 column we get $E - 1 = D$, and from the units column, $W \to E$. Substituting, as before, we get $2E - 2 \to E$, so $E = 2$ and $D = 1$. Now $21 \to N$, that is, $21 = N + c|B|$, where $c = 0$ or 1. Also $-c + N + 1 = I + |B|$, which demands that $N \geq |B|$, an impossibility. So there is no solution for any negative base.

II. *Comment by the Proposer.*

Easy, wasn't it?

Also solved for any positive base by CHARLES ASHBACHER, *Hiawatha, IA,* SEUNG-JIN BANG, *Seoul,* Korea, WILLIAM CHAU, *New York, NY,* HENRY S. LIEBERMAN, *Waban, MA,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* KENNETH M. WILKE, *Topeka, KS, and the* PROPOSER. *Base ten solutions were submitted by* JOHN T. ANNULIS, *University of Arkansas-Monticello,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* CAVELAND MATH GROUP, *Western Kentucky University, Bowling Green,* MARK EVANS, *Louisville, KY,* HOWARD FORMAN, *Parsippany, NJ,* DAWN M. GALAYDA, *St. Bonaventure University, NY,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* S. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* NATHAN JASPEN, *Stevens Institute of Technology, Hoboken, NJ,* LOWELL F. LYNDE, JR., *University of Arkansas at Monticello,* WADE H. SHERARD, *Furman University, Greenville, SC,* REX H. WU, *New York, NY, and* DAVID YAVENDITTI, *Alma, MI.*

746. [Spring 1991] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

Find the least positive integer n that will have remainder 1 when divided by r, the quotient will have remainder 2 when divided by r, the new quotient will have remainder 3 when divided by r, and so forth through $r - 1$ divisions. That is, $n = q_0$, and $q_{k-1} = q_k r + k$ for $k = 1, 2, \ldots, r - 1$, r a positive integer greater than 1.

I. *Solution by John Putz, Alma College, Alma, Michigan.*

We have $q_0 = 1 + q_1 r$, $q_1 = 2 + q_2 r$, ..., $q_{r-2} = r - 1 + q_{r-1} r$. Multiply the second equation by r, the third by r^2, and so forth, and then substitute to get

$$n = q_0 = 1 + 2r + 3r^2 + \ldots + (r-1)r^{r-2} + q_{r-1}r^{r-1}.$$

To minimize n, we choose $q_{r-1} = 0$ since $r > 1$. So

$$n = 1 + 2r + 3r^2 + \ldots + (r-1)r^{r-2}$$

and

$$m = r + 2r^2 + 3r^3 + \ldots + (r-1)r^{r-1}.$$

Subtracting, we have

$$(1-r)n = 1 + r + r^2 + \ldots + r^{r-2} - (r-1)r^{r-1} = \frac{1 - r^{r-1}}{1 - r} + (1 - r)r^{r-1},$$

so

$$n = r^{r-1} - \frac{r^{r-1} - 1}{(r-1)^2}.$$

II. *Solution by Stephen I. Gendler, Clarion University of Pennsylvania, Clarion, Pennsylvania*

The numbers described seem to be no more than

$(r-1)\,(r-2)\,(r-3) \ldots (3)\,(2)\,(1)$ in base r,

where each pair of parentheses is a digit, since the repeated division is a method for changing bases. Such numbers appear in any positive base r, the smallest being $r = 2$, $n = 1$. A few larger examples are listed below.

r	n (base r)	n (base ten)
3	21	7
4	321	57
5	4321	586
6	54321	7456
7	654321	114381
8	7654321	2054353
9	87654321	42374116
10	987654321	987654321

III. Comment by the Proposer.

This problem *is* a variant of the ordinary cocoanut-monkey problem in which the cocoanuts in a given pile are to be divided equally among r people the next morning. During the night one of the people sneaks out to the pile and divides it into r equal piles with exactly s cocoanuts left over, where $1 \leq s < r$. The s cocoanuts are thrown to a waiting monkey and the person hides one of the equal piles as his/her share. The remaining cocoanuts are repiled into one pile. As the night progresses, each of the r people in turn sneaks out to the pile and repeats the procedure of dividing the cocoanuts into r equal piles with exactly s cocoanuts left over, throwing the s cocoanuts to the monkey, and hiding one pile as that person's share. In the morning, there remain just enough cocoanuts to be divided equally among the r people. Here s cocoanuts plus $1/r$ of the remaining pile are removed exactly r times to leave a multiple of r cocoanuts in the pile. See problem 3242 in *The American Mathematical Monthly* (January 1928).

Also solved by CHARLES ASHBACHER, *Hiawatha, IA,* SEUNG-JIN BANG, *Seoul,* Korea, JAMES E. CAMPBELL, *Indiana University* at *Bloomington,* CAVELAND MATH GROUP, *Western Kentucky University. Bowling Green,* WILLIAM CHAU, *New York, NY,* CHARLES R. DIMINNIE, *St. Bonaventure University, NY,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* HOWARD FORMAN, *Parsippany, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* MOHAMMAD P. SHAIKH, *Western Michigan University, Kalamazoo, KENNETH M.* WILKE, *Topeka, KS, REX* H. WU, *New York, NY,* DAVID YAVENDITTI, *Alma, MI, and the* PROPOSER.

747. [Spring **1991**] ***Proposed by the late Jack Garfunkel, Flushing, New York.***

Let ABC be a triangle with inscribed circle (I) and let the line segments AI, BI, and CI cut the incircle at A', B', and C' respectively. Prove that

$$\sin A' + \sin B' + \sin C' \geq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2},$$

where A', B', and C' are the angles of triangle $A'B'C'$.

Solution by William Chau, New York, New York.

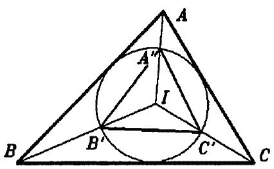
Since $2\angle A' = \angle BIC = \pi - (\angle B/2 + \angle C/2)$, it follows that

$$\sin 2A' = \sin\frac{B+C}{2} = \sin\frac{\pi - A}{2} = \cos\frac{A}{2},$$

with similar equalities for B and for C. Now the stated equation is equivalent to

$$\sin A' + \sin B' + \sin C' \geq \sin 2A' + \sin 2B' + \sin 2C',$$

which *is* item 2.4 on p. 18 of O. Bottema et al, *Geometric Inequalities,* Wolters-Noordhoff, Gronigen, *1968.*



Also solved by SCOTT H. BROWN, *Stuart Middle School, FL,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MURRAY S. KLAMKIN, *University of Alberta, Canada.* YOSHINOBU MURAYOSHI, *Eugene, OR,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* REX *H.* WU, *New York, NY, and the* PROPOSER, *Prielipp pointed out that this same problem appeared with his solution as problem 4274 in the March 1991* School Science and Mathematics.

748. [Spring **1991**] ***Proposed by the late John Howell, Littlerock, California.***

a) An urn contains n balls numbered 1 to n. Algernon, Beauregard, and Chauncey draw a ball one after another with replacement. The game is terminated when two consecutive drawings produce the same ball. Find the probabilities of terminating on Algernon's draw, on Beauregard's draw, and on Chauncey's draw.

b) Repeat the problem for the case that the game terminates when three consecutive drawings produce the same ball.

Amalgam of solutions by David Yavenditti, Alma, Michigan, and Morris Katz, Macwahoc, Maine.

Instead, we generalize the result for k consecutive drawings of the same ball, $k > 1$. We denote by $P(X)$ the probability the game terminates on X's draw. Let the players be denoted by X_i, where X_i = Algernon, Beauregard, or Chauncey according as $i \equiv 1$, 2, or 3 (mod 3). The first person who could win (terminate the game) is X_k on the kth play with probability $p = 1/n^{k-1}$. Let $q = 1 - p$. If X_k does not win at that turn, then X_{k+1} could win on the $(k+1)$st play with probability pq, or X_{k+2} could win on the $(k+2)$nd play with probability pq^2, etc. Hence

$$P(X_k) = p + pq^3 + pq^6 + pq^9 + \ldots$$

$$= \frac{p}{1-q^3} = \frac{1/n^{k-1}}{1-(1-1/n^{k-1})^3} = \frac{n^{2k-2}}{3n^{2k-2} - 3n^{k-1} + 1}.$$

Now, let X_k lose on the kth play (with probability q). Then X_{k+1} faces the same conditions that X_k did on the kth play, and we have

$$P(X_{k+1}) = q \cdot P(X_k) = \frac{n^{2k-2}(1 - 1/n^{k-1})}{3n^{2k-2} - 3n^{k-1} + 1} = \frac{n^{2k-2} - n^{k-1}}{3n^{2k-2} - 3n^{k-1} + 1}.$$

Similarly,

$$P(X_{k+2}) = q^2 \cdot P(X_k) = \frac{n^{2k-2}(1 - 1/n^{k-1})^2}{3n^{2k-2} - 3n^{k-1} + 1} = \frac{n^{2k-2} - 2n^{k-1} + 1}{3n^{2k-2} - 3n^{k-1} + 1}.$$

It follows that, for part (a),

$$P(B) = \frac{n^2}{3n^2 - 3n + 1}, \quad P(C) = \frac{n^2 - n}{3n^2 - 3n + 1}, \quad P(A) = \frac{n^2 - 2n + 1}{3n^2 - 3n + 1}.$$

Similarly, for part **(b)** we have that

$$P(C) = \frac{n^4}{3n^4 - 3n^2 + 1}, \quad P(A) = \frac{n^4 - n^2}{3n^4 - 3n^2 + 1}, \quad P(B) = \frac{n^4 - 2n^2 + 1}{3n^4 - 3n^2 + 1}$$

Also solved by CHARLES ASHBACHER, *(Part (a) only), Hiawatha, **IA,*** **JAMES** E. CAMPBELL, *(Part (a) only), Indiana University at **Bloomington,*** WILLIAM CHAU, *(Part (a) only). New **York, NY,*** **MARK** EVANS, *Louisville, **KY,*** HOWARD **FORMAN,** *(Part (a) only), **Parsippany, NJ,*** RICHARD *I.* **HESS,** *Rancho **Palos Verdes, CA,*** MOHAMMAD P. SHAIKH, *Western Michigan University, **Kalamazoo,*** REX **H. WU,** *(Part (a) only), New York, **NY,** and the* PROPOSER. *Not all solutions agreed with that of Morris Katz.*

749. [Spring **1991]** *Proposer* by *R. **S. Luthar,** University of Wisconsin Center at **Janesville, Janesville,** Wisconsin.*

If $\sin x + \sin y + \sin z = 0$, then prove that

$$|\sin 3x + \sin 3y + \sin 3z| \leq 12|xyz|.$$

*Solution by Bob **Prielipp,** University of Wisconsin-Oshkosh, **Oshkosh,** Wisconsin.*

If $a + b + c = 0$, then

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab) = 0.$$

Thus, since $\sin x + \sin y + \sin z = 0$, we have

$$\sin^3 x + \sin^3 y + \sin^3 z = 3 \sin x \sin y \sin z.$$

Also, since $\sin 3t = 3 \sin t - 4 \sin^3 t$, we get that

$$\sin 3x + \sin 3y + \sin 3z = -4(\sin^3 x + \sin^3 y + \sin^3 z) = -12 \sin x \sin y \sin z.$$

Hence the required inequality holds if and only if

$$|\sin x \sin y \sin z| \leq |xyz|,$$

which follows immediately from the well known inequality $|\sin t| \leq |t|$.

Also solved by SEUNG-JIN BANG, *Seoul,* Korea, MARTIN **BAZANT,** *Tucson, **AZ,*** WILLIAM CHAU, *New York, **NY,*** RUSSELL **EULER,** *Northwest Missouri State University. **Maryville,*** MURRAY S. **KLAMKIN,** *University of Alberta, Canada,* YOSHINOBU MURAYOSHI, ***Eugene, OR,*** MOHAMMAD P. SHAIKH, *Western Michigan University. **Kalamazoo,*** REX H. **WU,** *New York, NY, and the* PROPOSER.

750.** [Spring **1991]** *Proposed* by ***Dmitry** P. Mavlo, Moscow, **U.S.S.R.

Solve the system of equations

$$2^x y + (3^x)\sqrt{1 - y^2} = \sqrt{3} \quad \text{and} \quad 3^x y - (2^x)\sqrt{1 - y^2} = \sqrt{2}.$$

This problem appeared in the SYMP-86 Entrance Exam Mathematical Problems.

Solution by ***Parush Saxena,** Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

Letting $y = \sin \theta$, we must solve the equations

$$2^x \sin \theta + 3^x \cos \theta = \sqrt{3} \quad \text{and} \quad 3^x \sin \theta - 2^x \cos \theta = \sqrt{2}.$$

Now square these equations and add to get $2^{2x} + 3^{2x} = 5$, which has the unique solution $x = 1/2$ since the left side of the equation is an increasing function of x. Substituting $x = 1/2$ into the first of the original equations yields

$$\sqrt{2}\,y + \sqrt{3}\sqrt{1 - y^2} = \sqrt{3},$$

which, upon squaring and simplifying, reduces to

$$y(5y - 2\sqrt{6}) = 0, \text{ so } y = 0 \text{ or } y = \frac{2\sqrt{6}}{5}.$$

Only the latter value checks in the given equations, so the unique solution is $x = 1/2$ and $y = 2\sqrt{6}/5$.

Also solved by CHARLES ASHBACHER, *Hiawatha, **IA,*** SEUNG-JIN BANG, *Seoul,* Korea. FRANK P. BATTLES, *Massachusetts **Maritime** Academy, Buzzards Bay,* BARRY BRUNSON, *Western Kentucky University, Bowling Green,* JAMES E. CAMPBELL, *Indiana University at **Bloomington,*** **WILLIAM** CHAU, *New York, **NY,*** CHARLES **R. DIMINNIE,** *St. **Bonaventure University,** NY,* ROBERT **O. DOWNES,** *Long Beach, **CA,*** **MARK** EVANS, *Louisville, **KY,*** HOWARD **FORMAN,** ***Parsippany,** NJ,* ROBERT C. GEBHARDT, ***Hopatcong, NJ,*** RICHARD I. **HESS,** *Rancho **Palos Verdes, CA,*** HENRY S. **LIEBERMAN,** ***Waban,*** MA, PETER **A. LINDSTROM,** *North Lake College, **Irving,** TX,* G. **MAVRIGIAN,** ***Youngstown** State University. OH,* YOSHINOBU MURAYOSHI, *Eugene, OR.* **WILLIAM** *H.* **PEIRCE,** ***Stonington, CT,*** GEORGE *W.* **RAINEY,** *Los **Angeles, CA,*** MOHAMMAD P. SHAIKH, *Western Michigan University, **Kalamazoo,*** DAVE SMITH, *Messiah College, **Grantham,** PA,* **REX** H. **WU,** *New York, **NY,** and* DAVID **YAVENDITTI,** *Alma, MI. One faulty solution was also submitted.*

751. [Spring **1991**] Proposed by Murray *S. Klamkin,* University of Alberta, Edmonton, Alberta, ***Canada.***

Determine all pairs of positive numbers x and y such that

$$9(x+y)+\frac{1}{x}+\frac{1}{y}\ge 10+\frac{x}{y}+\frac{y}{x}.$$

I. Solution by ***Seung-Jin*** Bane, Seoul, Republic of Korea

Multiply the stated inequality by the positive quantity xy and rearrange the result to get the equivalent inequality

(1) $$(9y-1)x^2+(9y-1)(y-1)x+y(1-y)\ge 0.$$

Case 1. The left side is a quadratic polynomial whose discriminant is

$$D=(9y-1)(y-1)(3y-1)^2,$$

so $D\le 0$ for $1/9\le y\le 1$, whence the original inequality holds for all $x>0$ when $1/9\le y\le 1$. By the symmetry of the original inequality, it holds also for all $y>0$ when $1/9\le x\le 1$.

Case 2. Since Inequality (1) *can* be rewritten in the form

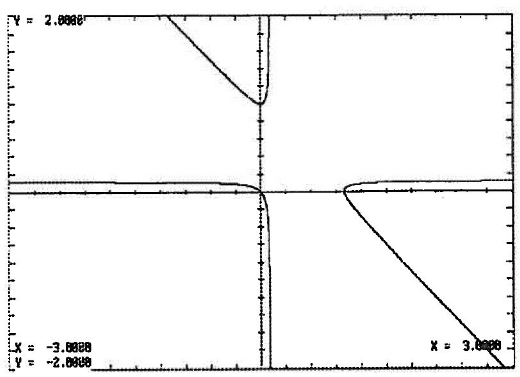
$$(1-9y)x(1-x)+(1-9x)y(1-y)+8xy\ge 0,$$

we see it is true for $0<x<1/9$ and $0<y<1/9$, and also for $x>1$ and $y>1$.

Case 3. We need only consider the region $0<x<1/9$ and $y>1$ and by symmetry the region $0<y<1/9$ and $x>1$. Apply the quadratic formula to the quadratic polynomial of Inequality (1) to get that, when $y>1$, we must have

$$x>\frac{-(9y-1)(y-1)+(3y-1)\sqrt{(9y-1)(y-1)}}{2(9y-1)}.$$

Now interchange x and y to get the corresponding inequality for the region where $x>1$.



II. Solution by Robert *C.* Gebhardt, ***Hopatcong,*** New Jersey.

Multiply the inequality by xy and rearrange to get

$$9x^2y+9xy^2+x+y-x^2-y^2-10xy\ge 0.$$

We graph the equation, as shown in the accompanying figure. The **curve** has intercepts **(0,0), (1,0), and (0,1).** Then any address in the first quadrant between the curves **(e.g. (1,1)** or **(0.321, 65.432))** will satisfy the given inequality.

III. Comment by Elizabeth ***Andy, Limerick,*** Maine.

The graph in the accompanying figure clearly shows asymptotes $x=1/9$, $y=1/9$, and $x+y=8/9$. Replace the original inequality by equality and rewrite it in the **form**

$$9(x+y)=10+\frac{x-1}{y}+\frac{y-1}{x}$$

and **finally**

$$9(x+y)=8+\frac{(x+y)(x+y-1)}{xy}.$$

We see that, when $x=-y+1$ and $|x|$ is large, **the** fraction is approximately zero, and we get the equation of the oblique asymptote $x+y=8/9$. Similarly, the equation *can* be rewritten in the form

$$9y=1+\frac{(9y-1)(1-x)}{y}+\frac{y-1}{x}.$$

If $|x|$ is large and $y=1/9$, then both fractions on the right are approximately zero, and the equation reduces to the horizontal asymptote $9y=1$. Thus we get the asymptotes algebraically, too.

Furthermore the original inequality shows there is symmetry in x and y, that is, in the **line** $y=x$. Therefore, in Inequality (1) of Solution I above, replace the inequality by equality, interchange x and y, and apply the quadratic formula to get that

$$y=\frac{-(9x-1)(x-1)\pm\sqrt{(9x-1)(x-1)(3x-1)^2}}{2(9x-1)},$$

which reduces to

$$y=-\frac{x-1}{2}\pm\frac{3x-1}{2}\sqrt{\frac{x-1}{9x-1}}.$$

When $|x|$ is large, then the quantity in the radical is approximately **1/9,** so the solution becomes

$$y\approx-\frac{x-1}{2}\pm\frac{3x-1}{6},$$

that is,

$$y = \frac{1}{3} \quad \text{or} \quad y = -x + \frac{2}{3}.$$

These are **not** the equations of the asymptotes! What is wrong? Why does the application of the quadratic-formula show incorrect asymptotes?

When a problem is most paradoxical,
*Then don't let it become **cardiotoxical.***
You just say it aloud
*To the P **M** E crowd.*
And the answer you'll get from some foxy gal.

Also solved by **WILLIAM** CHAU, *New York,* ***NY,*** **MARK** EVANS, *Louisville,* ***KY,*** RICHARD *I.* HESS, *Rancho* ***Palos*** *Verdes, CA, and the PROPOSER Two other solvers sent in a* ***pair*** *of faulty solutions and a* ***partial*** *solution.*

752. [Spring **1991]** *Proposed by the late Charles W.* ***Trigg,*** *San* ***Diego,*** *California.*

Martin Gardner ("Mathematical Games," *Scientific American,* **April** 1964, page 135) has shown that the minimum sum of three 3-digit primes that contain the nine non-zero digits is 999. Bind a set of **three** such primes that sums to another multiple of 37.

Solution by William Chau, New York, New York.

Let S be the sum of the three primes. The units digit of each prime is one of **1, 3, 7,** and 9, so **the** units digit of S *will* be **1, 3, 7,** or 9 according as **9, 7, 3,** or 1 is not used as the units digit of one of **the** primes. The maximum value of S is $100(6 + 8 + 9) + 10(2 + 4 + 5) + (1 + 3 + 7) = 2421 < 66\cdot37$ and the minimum value is 999.

Each prime is congruent to the sum of its digits modulo 9. Therefore $S \equiv 1 + 2 + 3 + \ldots + 9 = 45 \equiv 0 \pmod 9$, so S is a multiple of $9\cdot37 = 333$. The only multiple of 333 between 999 and 2421 that terminates in **1, 3, 7,** or 9 is 2331, so $S = 2331$ and the units digits of the primes are **1,** 3, and 7. For S to be that large, the hundreds digits of the primes must be **9, 8,** and **4** or 5 or 6. Since their tens digits are then 2, and two of 4 and 5 and 6, the tens column produces a carry of exactly **1** to the hundreds column, so the hundreds digits of the primes are **9, 8,** and 5. The tens digits are thus **2, 4,** and 6. From a table of primes we see that there are just 12 primes that meet the requirements for the addends: 521,523,541,547,563,821,823,827,863,941,947, and 967. It is easy now to find that there are exactly the four solutions **{521, 863, 947}, {541, 823, 967}, {563, 821, 947},** and **{563, 827, 941}.**

Also solved by CHARLES ASHBACHER, *Hiawatha,* ***IA,*** **MARK** EVANS, *Louisville,* ***KY,*** VICTOR *G.* **FESER,** *University of Mary, Bismarck, ND,* DAWN M. GALAYDA, *St. Bonaventure University,* ***NY,*** STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD *I.* **HESS,** *Rancho Palm Verdes,* ***CA,*** HENRY S. LIEBERMAN, *Waban,* MA, BOB PRIELIPP, *University of* ***Wisconsin-Oshkosh,*** KENNETH M. **WILKE,** *Topeka,* ***KS,*** REX **H.** WU, *New York, NY, and the* **PROPOSER.** ***Ashbacher,*** *Evans, Lieberman, and* ***Wu*** *each found the solution to the original problem* $149 + 263 + 587 = 999$.

753. [Spring **1991]** *Proposed by* ***R.*** *S. Luthar, University of Wisconsin Center at* ***Janesville, Janesville,*** *Wisconsin.*

Solve simultaneously

$$e^{4x} + e^{4y} = 82 \quad \text{and} \quad e^x \cdot e^y = 2$$

Solution ***by*** *George P.* ***Evanovich,*** *Saint Peter's College, Jersey City, New Jersey.*

Let $e^x = u + v$ and $e^y = u - v$. From the second given equation, $v = 1$, so we have

$$(u + 1)^4 + (u - 1)^4 = 82, \quad \text{or} \quad u^4 + 6u^2 - 40 = 0,$$

and hence $u = \pm 2$ or $\pm i\sqrt{10}$. Then $(e^x, e^y) = (3, 1), (-1, -3), (1 + i\sqrt{10}, -1 + i\sqrt{10})$, or $(1 - i\sqrt{10}, -1 - i\sqrt{10})$. Now $(x, y) = (\ln 3, 0)$ (the only real solution), $(\pi i + 2\pi k, \ln 3 + 2\pi k i)$, or $(\ln\sqrt{11} + i(\tan^{-1}(\pm\sqrt{10}) + 2\pi k), \ln\sqrt{11} + i(\tan^{-1}(\mp\sqrt{10}) + 2\pi k))$, where k is an integer.

Also solved by JOHN T. **ANNULIS,** *University of Arkansas-Monticello,* CHARLES **ASHBACHER,** *Hiawatha,* ***IA,*** **SEUNG-JIN** BANG, *Seoul,* Korea, FRANK P. BATTLES, *Massachusetts* ***Maritime*** *Academy,* ***Buzzards*** *Bay,* DIETER BENNEWITZ, ***Koblenz,*** *Germany,* SCOTT *H.* BROWN, *Stuart Middle School,* ***FL,*** BARRY BRUNSON, *Western Kentucky University, Bowling Green,* JAMES E. CAMPBELL, *Indiana University at* ***Bloomington,*** **CAVELAND** MATH GROUP, *Western Kentucky University, Bowling Grew,* JEAN CHAPMAN, ***Creston,*** ***IA,*** WILLIAM CHAU, *New York.* ***NY,*** PATRICK COSTELLO, *Eastern Kentucky University, Richmond,* CHARLES R. **DIMINNIE,** *St. Bonaventure University,* ***NY,*** ROBERT **O. DOWNES,** *Long Beach,* CA, RUSSELL **EULER,** *Northwest* ***Missouri*** *State University,* ***Maryville,*** **MARK** EVANS, *Louisville.* ***KY,*** VICTOR G. **FESER,** *University of Mary, Bismarck, ND,* HOWARD **FORMAN,** ***Parsippany,*** *NJ,* DAWN *M.* **GALAYDA,** *St. Bonaventure University,* ***NY,*** ROBERT C. GEBHARDT, *Hopatcong,* ***NJ,*** STEPHEN I. **GENDLER,** *Clarion University of Pennsylvania,* RICHARD A. **GIBBS,** *Fort Lewis College,* ***Durango,*** *CO,* STAN **HARTZLER,** *Messiah College,* ***Grantham,*** *PA,* RICHARD I. HESS, *Rancho Palos Verdes,* ***CA,*** NATHAN JASPEN, *Stevens Institute of Technology,* ***Hoboken,*** ***NJ,*** HENRY S. UEBERMAN, *Waban,* AM, PETER A. LINDSTROM, *North Lake College, Irving,* ***TX,*** LOWELL F. **LYNDE,** JR., *University of Arkansas at Monticello,* G. **MAVRIGIAN,** *Youngstown State University, OH,* YOSHINOBU MURAYOSHI, *Eugene,* OR, WILLIAM *H.* PEIRCE, ***Stonington,*** ***CT,*** BOB **PRIELIPP,** *University of* ***Wisconsin-Oshkosh,*** GEORGE W. **RAINEY,** *Los* ***Angeles,*** ***CA,*** **PARUSH** SAXENA, *Massachusetts Maritime Academy, Buzzards Bay,* **MOHAMMAD** P. SHAIKH, *Western Michigan University,* ***Kalamazoo,*** WADE *H.* **SHERARD,** ***Furman*** *University, Greenville,* ***SC,*** KENNETH M. WILKE, *Topeka,* ***KS,*** REX H. **WU,** *New* ***York,*** ***NY,*** DAVID **YAVENDITTI,** *Alma, MI. and the* PROPOSER.

754. [Spring **1991]** *Proposed by* ***Seung-Jin*** *Bang, Seoul, Korea.*

Let a, = a, = 1, $a_3 = 2$, and $a_{n+1} = a$, $\cdot\, a_{n-1} + a_{n-2}$ for $n > 3$. Show that

$$a_{n+2}a_na_{n-2} - a_{n+2}a_{n-1}^2 - a_{n+1}^2a_{n-2} + 2a_{n+1}a_na_{n-1} - a_n^3 + 3 = 0.$$

I. *Solution by the Proposer.*

It suffices to show that

$$\begin{vmatrix} a_{n+2} & a_{n+1} & a_n \\ a_{n+1} & a_n & a_{n-1} \\ a_n & a_{n-1} & a_{n-2} \end{vmatrix} = -3.$$

To that end, we have

$$\begin{bmatrix} a_{n+2} & a_{n+1} & a_n \\ a_{n+1} & a_n & a_{n-1} \\ a_n & a_{n-1} & a_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} & a_n & a_{n-1} \\ a_n & a_{n-1} & a_{n-2} \\ a_{n-1} & a_{n-2} & a_{n-3} \end{bmatrix},$$

and hence

$$\begin{bmatrix} a_{n+2} & a_{n+1} & a_n \\ a_{n+1} & a_n & a_{n-1} \\ a_n & a_{n-1} & a_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Since the two matrices on the right side of this last equation have determinants 1 and -3 respectively, the determinant of the matrix on the left side is -3.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

As stated, a_4 is not determined, so that the given equation is not necessarily true for $n = 3$. If the recursion formula is valid for $n \geq 3$, then $a_4 = a_3 - a_2 + a_1 = 2$ and $a_5 = 1$. Since we now have $a_{n+2} = a_{n-2}$ for $n \geq 3$, it suffices to check the desired equation for $n = 3, 4, 5$, and 6, for which values it is true.

III. *Solution by Rex H. Wu, New York, New York*

To express a_n in terms of n, let $a_n = \lambda$. Then

$$\lambda^{n+1} = \lambda^n - \lambda^{n-1} + \lambda^{n-2}, \text{ so that } \lambda^3 = \lambda^2 - \lambda + 1,$$

$$(\lambda - 1)(\lambda^2 + 1) = 0, \text{ and finally } \lambda = 1, \pm i.$$

Any linear combination of solutions is also a solution, so we have that

$$a_n = \alpha + \beta i^n + \gamma(-i)^n$$

for some complex constants α, β, and γ. By hypothesis we must have

$$a_1 = \alpha + \beta i - \gamma i = 1, \quad a_2 = \alpha - \beta - \gamma = 1,$$

$$\text{and } a_3 = \alpha - \beta i + \gamma i = 2,$$

which we solve simultaneously to get that

$$\alpha = \frac{3}{2}, \quad \beta = \frac{i-1}{4i}, \quad \text{and} \quad \gamma = \frac{i+1}{4i}.$$

Therefore,

$$a_n = \frac{3}{2} + \frac{i-1}{4i} i^n + \frac{i+1}{4i} (-i)^n.$$

It follows immediately that $a_{n+4} = a_n$. By tedious but straightforward algebra one can show that $a_n \cdot a_{n+2} = 2$ and that $a_n^2 + a_{n+2}^2 = 5$. Now we have

$$\begin{aligned} & a_{n+2}a_n a_{n-2} - a_{n+2}a_{n-1}^2 - a_{n+1}^2 a_{n-2} + 2a_{n+1}a_n a_{n-1} - a_n^3 + 3 \\ &= 2a_{n-2} - a_{n-2}a_{n-1}^2 - a_{n+1}^2 a_{n-2} + 4a_n - a_n^3 + 3 \\ &= 2a_{n-2} - 5a_{n-2} + 4a_n - a_n^3 + 3 \\ &= -3a_{n-2} + 4a_n - a_n^3 + 3. \end{aligned}$$

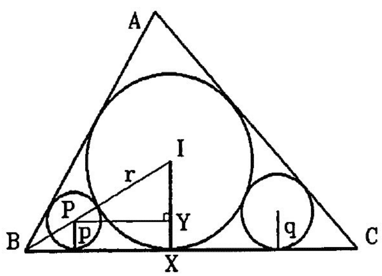
Since $a_n = 1$ if $n \equiv 1$ or $2 \pmod 4$ and $a_n = 2$ if $n \equiv 0$ or $3 \pmod 4$, we need verify only that the last displayed line is zero for $a_{n-2} = 1$ and $a_n = 2$, and for $a_{n-2} = 2$ and $a_n = 1$, which is easily accomplished.

Editor's comment. Only Klamkin spotted the omission, which was my error. The proposer had stated only the defining equations; I added the inequality. So I shall do my penance at least $n > 3$ times.

Also solved by CHARLES ASHBACHER, *Hiawatha, IA,* SCOTT H. BROWN, *Stuart Middle School, FL,* JAMES E. CAMPBELL, *Indiana University at Bloomington,* CAVELAND MATH GROUP, *Western Kentucky University, Bowling Green,* WILLIAM CHAU, *New York, NY,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* HOWARD FORMAN, *Parsippany, NJ,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* WILLIAM H. PEIRCE, *Stonington, CT,* MOHAMMAD P. SHAIKH, *Western Michigan University, Kalamazoo,* KENNETH M. WILKE, *Topeka, KS, and* DAVID YAVENDITTI, *Alma, MI.*

755. [Spring 1991] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

In triangle ABC, a circle of radius p is inscribed in the wedge bounded by sides AB and BC and the incircle (I) of the triangle. A circle of radius q is inscribed in the wedge bounded by sides AC and BC and the incircle. If $p = q$, prove that $AB = AC$.



I. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

Let the incircle (I) touch BC at X. If the two side circles have the same radius, then a reflection about the line IX leaves the picture unchanged, whence $AB = AC$.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Draw the angle bisector BI of angle B and let r denote the **inradius.** Let the parallel to BC through the center P of the circle of radius p cut IX at Y. Then $IY = r - p$ and $ZP = r + p$, and it now follows easily that

$$\sin\frac{B}{2} = \frac{r-p}{r+p} \quad \text{and similarly} \quad \sin\frac{C}{2} = \frac{r-q}{r+q}.$$

Finally, $p = q$ implies that $\sin(B/2) = \sin(C/2)$ and hence that $AB = AC$.

Also solved by SEUNG-JIN BANG, *Seoul, Korea,* DIETER BENNEWITZ, *Koblenz, Germany,* SCOTT H. BROWN, *Stuart Middle School, FL,* WILLIAM CHAU, *New York, NY,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* HENRY S. LIEBERMAN, *Waban,* MA, T. R. K. PAPPU, *Occidental College, Los Angeles, CA,* MOHAMMAD P. SHAIKH, *Western Michigan University, Kalamazoo,* WADE H. SHERARD, *Funnan University, Greenville, SC,* REX H. WU, *New York, NY,* DAVID YAVENDITTI, *Alma, MI, and the* PROPOSER.

756. [Spring **1991**] *Proposed by Basil Rennie, Burnside, South Australia.*

Consider covering the unit interval $[0,1]$ with n measurable subsets, under the constraint that all n subsets must have the same centroid. The centroid m of a set E may be defined by $\int_E (x - m)\,dx = 0$. How **can** you choose the n sets to minimize m?

For example, if $n = 4$, it is possible to make $m = 7/20$ by choosing the four sets $[0,2/5]\cup[9/10,1]$, $[0,1/5]\cup[4/5,9/10]$, $[1/20,1/4]\cup[7/10,4/5]$, and $[0,7/10]$.

Solution by the Proposer.

The smallest value of m is $1/(1 + \sqrt{n})$, which we denote by c. For, let E_r (for $r = 1, 2, ..., n$) consist of the union of the two intervals $[0, c]$ and $[c + c\sqrt{(r-1)}, c + c\sqrt{r}]$. Each E_r has centroid c and together they cover the interval.

To show this value c is best possible, take n sets E_r covering the interval and with centroids at m. Divide each set into E_r' to the left of m and E_r'' to the right of m. The first moments of the two subsets about m must add to zero, and therefore the moment of E_r'' about m can be no more **than** $m^2/2$, but the sum of all these moments over the n sets is at least $(1-m)^2/2$. Hence, $nm^2 \ge (1-m)^2$, or $[m(\sqrt{n} + 1) - 1][m(\sqrt{n} - 1) + 1] \ge 0$, which is true whenever the quantity in the first brackets is **nonnegative;** that is, when $m \ge c$.

757. [Spring **1991**] *Proposed by Paul Anthony Courtney, graduate student, San Diego State University, San Diego, California.*

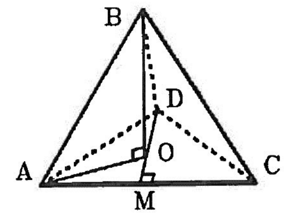
Find the overall height of the pyramid formed from four spherical balls of radius r. Student solutions are especially solicited.

Solution by David Yavenditti, high school student, Alma, Michigan.

Consider instead the pyramid formed by n triangular stacks of spheres, each of radius r. Let h be the overall height and let A, B, C, D be the centers of the four comer spheres, which determine a regular tetrahedron of edge $2r(n-1)$, as shown in the accompanying figure. We must find the length of the altitude BO of the tetrahedron, O being the center of the equilateral triangle ACD. Then OM is perpendicular to AC and triangle AOM is a $30°$-$60°$ right triangle, so $AO = 2r(n-1)/\sqrt{3}$. Since ABO is a right triangle, then $BO = 2r(n-1)\sqrt{6}/3$, and the overall height is given by

$$h = BO + 2r = 2r\left(1 + \frac{(n-1)\sqrt{6}}{3}\right).$$

The case we seek is $n = 2$, so $h = 2r(1 + \sqrt{6}/3)$.



Also solved by CHARLES ASHBACHER, *Hiawatha, IA,* MARTIN BAZANT, *Tucson, AZ,* WILLIAM CHAU, *New York, NY,* ROB DOWNES, *Long Beach, CA,* RUSSELL EULER, *Northwest Missouri Stale University, Maryville,* MARK EVANS, *Louisville, KY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* LOWELL F. LYNDE, JR., *University of Arkansas at Monticello,* MOHAMMAD P. SHAIKH, *Western Michigan University, Kalamazoo, REX H.* WU, *New York, NY, and the* PROPOSER.

MESSAGE FROM THE SECRETARY-TREASURER

Copies of the new, revised Constitution and Bylaws are now available. The prices are: *$1.50* for each of the first four copies and *$1* for each copy thereafter. **I.e., $(1.50** n**)** for $n < 4$ and **$(n+2)** for $n \geq 4$.

The videotape of Professor Joseph A. **Gallian's** AMS-MAA-PME Invited Address, "The Mathematics of Identification Numbers," given as part of **PME's** 75th Anniversary Celebration at Boulder, CO, in August, *1989,* is also still available. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of *$10.* Please contact *my* office if you desire to borrow the tape, telling me the date on which you would like to use it. I prefer to mail the tape directly to faculty advisors, and expect them to take responsibility for returning it to my office. Please submit your request in writing and include a phone number and a time that I might reach you if there are problems. Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC *27858.*

UPCOMING PI MU EPSILON 1992 NATIONAL MEETING

The national meeting for Pi Mu Epsilon this year will be very special. Usually, the national meeting is held in conjunction with the national meetings of the American Mathematical Society and the Mathematical Association of America. In 1992, however, the International Congress of Mathematics Educators (ICME) will hold its annual meeting in Quebec City, in Canada. It has been the policy of the AMS and MAA that in order to avoid a conflict in scheduling, summer meetings are not held in years when an international mathematics meeting (e.g., ICME or the International Congress of Mathematicians) takes place in North America. For this reason, there will be no AMS-MAA national meeting this summer.

Because of these special circumstances, Pi Mu Epsilon will hold its summer meeting in conjunction with the meeting of the MAA Student Chapters. The meeting will take place August 5-8, at Miami University in Oxford, Ohio.

The meeting will begin on the evening of Wednesday, August 5, with a Student Pizza Party and Reception. (Registration and room check-in will begin in the afternoon and continue throughout the evening.)

Highlights of Thursday's program (August 6) will include the MAA Invited Lecture, by Peter Hilton; a reception for Professor Hilton; sessions for contributed papers by PME and MAA student chapter members; presentations by the MAA Modeling Contest winners; meetings of the PME Council and the MAA Student Chapter Committee; and an excursion to the nearby King's Island Theme Park.

The program on Friday, August 7, will feature more student presentations; a choice of two minicourses (open to students and faculty); a panel discussion and display on careers; the Pi Mu Epsilon Banquet; and, finally, the J. Sutherland Frame lecture. This year's Frame lecturer will be Underwood Dudley. The Pi Mu Epsilon portion of the meetings will conclude with informal gatherings after the Frame Lecture.

The meeting will conclude on Saturday, August 8, with the final session of MAA student papers and a choice of two minicourses.

TRAVEL SUPPORT FOR THE SUMMER MEETING

Pi Mu Epsilon will provide travel support for one student speaker from each chapter. If a chapter is not represented by a student speaker, Pi Mu Epsilon will provide one-half support for a student delegate. Full support is defined to be full round-trip air fare (including ground transportation) from the student's school or home to Oxford, Ohio, up to a maximum of $600. (Delegates will receive up to $300.) A student who chooses to drive will receive 25 cents per mile for the round trip from school or home to Oxford, up to $600. (Delegates will receive $12\frac{1}{2}$ cents per mile, up to $300. Travel support will be provided for only one student per chapter. However, if several students from the same chapter wish to attend, they may share the travel support, if they choose to do so. (Special discounted group airline tickets are available on Delta Airlines through Travel Unlimited, the official travel agency for the conference: 1-800-466-7555.)

For further information about the meeting and the travel support:

SEE YOUR PI MU EPSILON ADVISOR

GLEANINGS FROM THE CHAPTER REPORTS

CONNECTICUT GAMMA (Fairfield University) During the fall semester the chapter sponsored the second annual Math Bowl Contest. Eight teams of four students competed in a "GE College Bowl" type of competition, in which all the questions were mathematical. In the spring, members of Pi Mu Epsilon assisted the Mathematics Department in coordinating the activities for Math Counts, which is a mathematics contest for junior high school students. At the annual spring initiation ceremony thirty-two new members were inducted. "Biostatistics: Who, What, Why, and When?" by Kerrie Eileen Boyle of the Research Triangle Institute was the title of the Pi Mu Epsilon Lecture during the induction ceremony. Dr. Boyle, a 1974 graduate of Fairfield, was also inducted. During the Annual Arts and Sciences Awards Ceremony, two members, Thomas Lipka and Francis Maurais received recognition for their outstanding performance in mathematics. Each was given a Pi Mu Epsilon certificate of achievement, a book each selected in an area of mathematics, and a one-year membership in the Mathematical Association of America.

INDIANA GAMMA (Rose-Hulman Institute of Technology) In the fall of 1990, six students attended the Miami University Conference, with John O'Bryan, Jeff Dierckman, and Omar Zaidi presenting papers.

The chapter helped administer the RHIT–St. Mary of the Woods Mathematics Competition (for area high school students) and the 2nd Annual Alfred R. Schmidt Freshmen Mathematics Competition at Rose-Hulman. Mark Roseberry took first place and Jonathan Atkins took second. Our chapter helped the Rose-Hulman Mathematics Department stage the Annual Rose-Hulman Undergraduate Mathematics Conference, which involved over 80 participants and 25 papers. Seven of our students gave papers: John O'Bryan, Jeff Dierckman, Omar Zaidi, Mark Roseberry, Jonathan Atkins, Ben Nicholson, and Tony Hinrichs. Five teams of three members each participated in the Indiana College Mathematics Competition, with the RHIT team of John O'Bryan, Mark Roseberry, and Jonathan Atkins taking first place.

On April 24, 32 new members were initiated into the Indiana Gamma Chapter. It was the 25th anniversary of the founding of the Chapter. The speaker at our initiation banquet was Dr. David Ballew, President of Pi Mu Epsilon and Chairman of the Computer Science Department at Western Illinois University.

KANSAS GAMMA (The Wichita State University) The chapter had a number of speakers during the year. The speakers were: Joseph Stafford, "Tilings," Abdelmalek Kemmou, "Fuzzy Set Theory & the Logic of the Continuum;" Ming Liu "Design of Experiment;" Rajiv Bagai, "Formal Logic as a Programming Language;" Dewi Saleh, "Some Mathematical Puzzles." Members of the chapter held free help sessions for undergraduate courses. One of the members, Abdelmalek Kemmou, gave a talk at the joint meetings of the Kansas Section of the MAA and the Kansas Association of Teachers in Mathematics, held at Southwestern College in April, 1991. The chapter also started a publication, called ALEPH TWO. The publication is intended to contain mathematical investigations, mostly by students.

MINNESOTA ZETA (St. Mary's College) The Chapter conducted a number of mathematics colloquia and several chapter-wide business meetings. The Chapter celebrated Math Awareness Week in April with two main activities: Professor Ken Kasin, of St. Mary's presented a talk entitled "Elementary Concepts of Mathematical Chaos"; and eleven new members were initiated into the chapter.

OHIO ZETA (University of Dayton) The chapter continued to be active this year. Among other activities, the members presented several talks at various meetings and conferences. Five students presented talks at the Pi Mu Epsilon Meeting in Columbus, Ohio, in August. Four of them presented the results of the research they conducted in the program "Research Experiences for Undergraduates in Algebraic Graph Theory at the University of Dayton." This program was sponsored by the NSF and Professors **Higgins** and **Mushenheim** conducted the program during the summer of 1990. All of these five students also gave talks at the Pi Mu Epsilon Regional Conference held at Miami University, Oxford, Ohio, in September, 1990. These students are Marjorie August, David Gebhard, Tom Bohman, **Chicako** Mese, and Colleen Hoover. David Gebhard and David **Kaas** presented talks at the Spring Meeting of the Ohio Section of the MAA held at Bowling Green, Ohio, in April, 1991.

Chicako Mese, Tom Bohman, and Colleen Hoover jointly received **UD's** Faculty Award for Excellence in Mathematics, while Kristen Toft and Kristine **Fromm** shared this year's Sophomore Class Award.

VIRGINIA ALPHA (University of Richmond) In the fall, in addition to an initiation ceremony, the Chapter co-sponsored a **Math/Computer** Science Department colloquium on October 22. The speaker, Professor Jim Kuzmanovich, from Wake Forest University, spoke on "The Lore of Infinity." In the spring, the Chapter held a research forum where four student members, who were engaged in independent research projects, gave 15-minute talks on their projects and how they got started. The speakers were **Fran** Centofante, Jeff Michel, John Murphy, and David Flader. David Flader presented his paper on Game Theory and Pseudo-Boolean Functions at the National Pi Mu Epsilon Meeting in Orono, Maine, in August. The final event of the year was the annual Pi Mu Epsilon picnic (co-sponsored with the Computer Science Club). At this picnic, Jeff **Michel** was presented with the award for Outstanding Computer Science Student and David Flader was presented with the award for Outstanding Mathematics Student. Freshman. Kelly **Donnellon,** was presented with the Pi Mu Epsilon Book Award for outstanding work in Calculus I and **II.**

WISCONSIN DELTA (St. Norbert College) In August, 1990, three students attended the Pi Mu Epsilon National Conference in Columbus, Ohio. Amy Krebsbach, Mike Lang, and Dave **Ol-**son were in attendance, with Mike Lang presenting a paper. In April, 1991, Amy Krebsbach presented a paper at the St. John's University Regional Math Conference. Also in attendance were Dawn Boyung, Chris Cypcar, Amy **Gerrits,** Mike Lang, and Mike Zittlow.

SNC was host to several speakers during the year. Dr. Bill Shay (UW–Green Bay) spoke on "Cyclic Redundancy Check – Error Detection Using Polynomial Division." Other speakers were: Dr. Alan Parks (Lawrence University) on "Genetic Assembly Line Balancing" and Richard **Witalka** and John Towne (Schneider National Corporation) on 'The C Programming Language."

Perhaps the biggest event of the year for the chapter was hosting the Fifth Annual Pi Mu Epsilon Regional Conference in November. The featured speaker was Dr. Jeanne **LaDuke,** of **DePaul** University, who spoke about the role of women in American mathematics. Laura **Donzelli** of SNC presented one of the 14 student papers at the conference.

Other significant events included the Ninth Annual SNC High School Math Meet held in conjunction with **SNC's** math club, Sigma Nu Delta. The annual Brenda Roebke Volleyball Tournament was also held in cooperation with Sigma Nu Delta. Part of the proceeds from this tournament go toward a scholarship fund for SNC students majoring in mathematics. This year's winner was Linda **Mueller.**

IN MEMORIAM

John T. **O'Bryan,** the president of the Indiana Gamma Chapter of Pi Mu Epsilon, at **Rose-Hulman** Institute of Technology, died December 16, 1991, as a result of injuries he received in a car accident.

John was one of the most active and productive members the Indiana Gamma Chapter has ever had. Between **April, 1990,** and September, 1991, he presented five different papers at six different conferences and meetings. John's most outstanding work resulted from his participation in an **NSF-**funded **REU** project at Rose-Hulman, which he attended between his sophomore and junior years. His paper "Maximal Order Three-Rewriteable Subgroups of Symmetric Groups" became the initial Rose-Hulman Institute of **Technology** Technical Report. John also presented this paper at a special session during the 1991 Winter Meeting of the MAA held in San **Francisco.**

John's REU experience also led to a paper titled "Large 'Almost Abelian' Subgroups of the Symmetric Group," which he presented at the 17th Annual Regional Pi Mu Epsilon Meeting held at Miami University, Oxford, Ohio. This paper became part of a joint paper written with Dr. Gary Sherman, of Rose-Hulman, titled "Undergraduates, CALEY, and Mathematics," which has been submitted to the ***Journal of Technology in Mathematics.***

But John's work wasn't limited to pure mathematics. During the summer between his junior and senior year, John worked as a summer researcher, in applied mathematics, as a member of the Outstanding Student Summer Program sponsored by Sandia National Laboratories in New Mexico. This experience led to his paper "Parallelization of a Parameter Identification Problem," which he gave at the 18th Annual Pi Mu Epsilon Conference at Miami University, in September, 1991. This was to be John's final Pi Mu Epsilon paper.

As a **mathematics/physics** double major at **Rose-Hulman,** John participated in many mathematics competitions as a leading member of the Rose-Hulman team. As a scholar, he received numerous awards, including the top freshman mathematics award, the top freshman student award, the top sophomore student award, and the top junior student award. This spring he will be awarded, posthumously, the Clarence P. **Sousley** Award as "a graduating mathematics major who has demonstrated exceptional performance in his field.'

John truly lived by the Pi Mu Epsilon pledge "... I will exert my best efforts to promote true scholarship, particularly in mathematics; and that I will support the objectives of the Pi Mu Epsilon Society."

(Elton Graves, RHIT Mathematics Department)

NINETEENTH ANNUAL
PI MU EPSILON
STUDENT CONFERENCE
MIAMI UNIVERSITY
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Call for student papers and guests
Friday and Saturday
October 2 – 3, 1992
Held in conjunction with
The Conference on History of Mathematics
featuring
Judith *Grabiner*, *William Dunham*, Victor *Katz*, *Fredrick Rickey*

We invite you to join us. There will be sessions of the student *conference* on Friday evening and Saturday afternoon. Free overnight lodging for *all* students will be arranged with Miami students. Each student should bring a sleeping bag. All student guests are invited to a free *Friday* evening pizza party supper, and speakers will be treated to a Saturday noon picnic lunch. *Talks* may be on any topic related to mathematics, statistics or computing. We welcome items ranging from expository to research, interesting applications, problems, summer employment, etc. Presentation time should be *fifteen* or *thirty* minutes.

We need your title, presentation time (15 or 30 min.), preferred date (Fri. or Sat.) and a 50 (approx.) word abstract by September 25, 1992. Please send to

Professor Milton D. Cox
Department of Mathematics and Statistics
Miami University
Oxford, Ohio 45056

The History of Mathematics Conference
begins
Friday afternoon, October 2.
Contact us for more details.

ΠΜΕ

St. Norbert College

Seventh Annual

PI MU EPSILON

Regional Undergraduate Math Conference

October 30-31, 1992

Featured Speaker: Jim **Kasum**
Cardinal Stritch College

Sponsored by: St. Norbert College Chapter of ΠΜΕ
and
St. Norbert College SNA Math Club

The conference will begin on Friday evening and continue through Saturday noon. Highlights of the conference will include sessions for student papers and two presentations by Professor **Kasum**, one on Friday evening and one on Saturday morning. Anyone interested In undergraduate mathematics is welcome to attend. There is no registration fee.

For information, contact:
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ΣΝΔ

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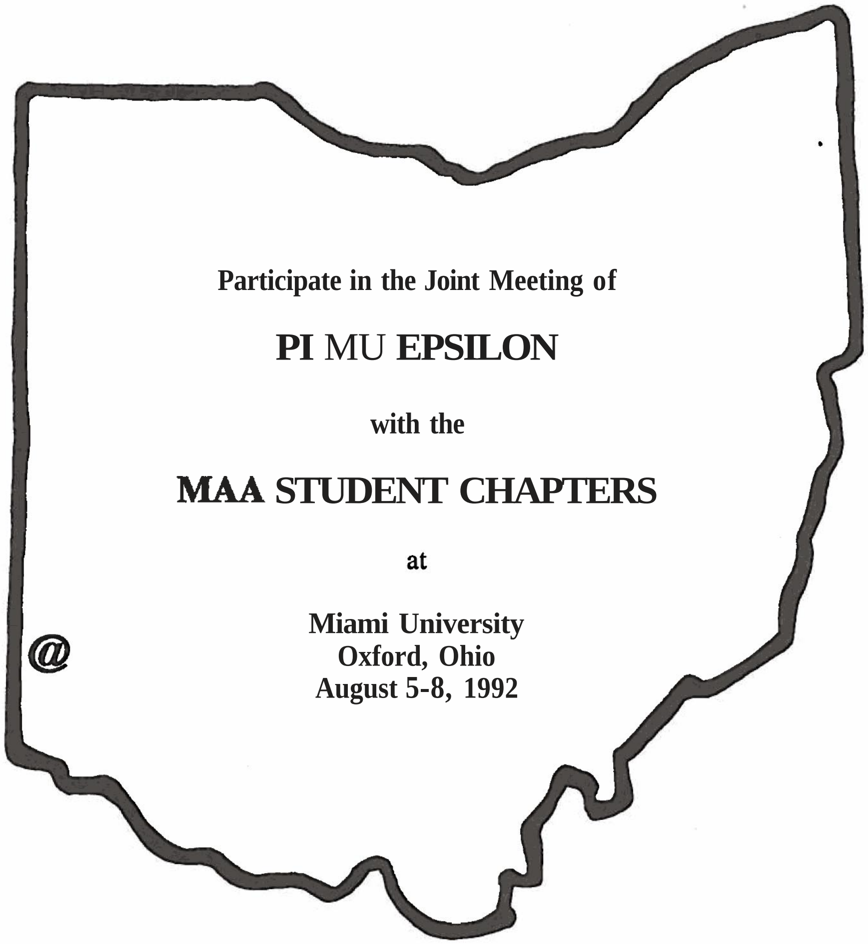

* See page 414 for farther details.

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