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CONTENTS

ORDERED PLANES

R.B. KILLGROVE

1. *Introduction.*

The author intends to write a text on geometry for future High School teachers, or perhaps for a one-year graduate course for High School teachers working toward a Master's Degree in the teaching of mathematics. The principal topics covered in the text will be Field Planes, Ordered Planes, and Finite Geometries. Field Planes having already been summarily covered in an earlier article [12], the present one is devoted to Ordered Planes. For additional information about these two topics, see references [7]-[11]. Although the author's specialty is using the computer to obtain results in Finite Geometries, he does not intend to discuss this topic here because of the very fine sources already available (see [2], [4], [6], and [13]).

2. *Ordered planes.*

For the sake of completeness, we repeat the definition of an ordered plane from [12].

Let Σ be a set of elements called *points*, and ω a ternary relation on Σ. If the ordered triple $(A,B,C) \in \omega$, then we will say that the points are *in the order* ABC and denote this by ωABC. If A and B are distinct points of Σ, then the *line* AB is the set of all points $P \in \Sigma$ such that

$$\omega PAB \quad \text{or} \quad P = A \quad \text{or} \quad \omega APB \quad \text{or} \quad P = B \quad \text{or} \quad \omega ABP.$$

(Note that every line has at least two distinct points by definition, since "line AB" implies $A \neq B$.) The pair (Σ,ω) is then called an *ordered plane* if the following six axioms are satisfied:

O_1. If ωABC, then A,B,C are all distinct.

O_2. If ωABC, then not ωBCA.

O_3. If $A \neq B$, then there is at least one point C such that ωABC.

O_4. There are three noncollinear points.

O_5. If C and D are distinct points of line AB, then A is on line CD.

O_6. (The Pasch axiom) If A,B,C are noncollinear points, if D is a point such that ωADB, and if there is a line $k \neq AB$ such that $D \in k$ and $C \notin k$, then there is a point E such that $E \in k$ and ωAEC or ωBEC.

3. *Elementary theorems on ordered planes.*

We now list a few of the most fundamental theorems on ordered planes. We do not include the proofs. Many institutions give an R.L. Moore type of course starting with O_1-O_6 or an equivalent set of axioms, and the students have to come up with all the proofs. The reader can do the same or else look up the proofs in Forder [5]. We will in a couple of places add a comment pointing out some pitfalls to be avoided by the do-it-yourselfers.

Order Theorem. If ωABC, then ωCBA but none of ωBCA, ωCAB, ωBAC, ωACB.

Join Theorem. For distinct points A and B, there is one and only one line k such that A $\in k$ and B $\in k$. (k is said to *join* A and B.)

Transversal Theorem. If A $\notin$ line BC, if ωBCD, and if ωAEC, then there is a point F such that ωFED and ωAFB. (The transversal is DE.)

Fano's Theorem. If A,B,C are noncollinear, and if ωAB'C, ωBC'A, ωCA'B, then A',B',C' are also noncollinear.

Betweenness Theorem. For distinct points A and B, there is at least one point M such that ωAMB.

Sometimes students come up with ostensibly correct proofs of the Betweenness Theorem which do not use the Pasch axiom or the Transversal Theorem. That these "proofs" are invalid is shown by the *integer plane*, which consists of all points with integer coordinates and all lines of the rational plane joining at least two such points. Axioms O_1-O_5 hold in the integer plane, while the Pasch axiom, the Transversal Theorem, and the Betweenness Theorem do not.

Infinitude Theorem. There are infinitely many points on a line.

A correct proof of the Infinitude Theorem requires the use of some lemmas such as: if ωABC and ωBCD, then ωACD. At least one of these lemmas requires the use of the Pasch axiom or the Transversal Theorem. When these lemmas are omitted, as they sometimes are in student proofs, the Infinitude Theorem appears to be a consequence of only O_1-O_5 and the Betweenness Theorem. That this is incorrect is shown by the counterexample of the field plane over the field of residues modulo 5, for which O_1-O_5 and the Betweenness Theorem hold but not the Infinitude Theorem since every line has exactly 5 points. In this plane, we take ωABC to mean that B is the midpoint of AC. Thus if A:(a,b) and C:(c,d), then ωABC if and only if B:$(3a+3c,3b+3d)$ since 3 is the multiplicative inverse of 2.

The following definitions will now be useful. A *closed interval* [AB] is the set of all points P $\in \Sigma$ such that P = A or P = B or ωAPB (when A $\neq$ B, with A = B permitted). Then A and B are the *endpoints* of [AB] and the other points, if any, are its *interior points*. If A $\neq$ B, the *open interval* (AB) is the set of interior points of [AB]. To say that a set S is *convex* means that [AB] is contained in S for every A,B $\in S$. It is an easy corollary that every line is a convex set.

Interval Theorem. Every interval AB, closed or open, is convex.

Continuing with the definitions, two intervals (closed or open) *overlap* if and only if their intersection contains at least two points. If two intervals are not disjoint and do not overlap, then they *cross*, unless the common point is an endpoint, in which case they *abut*.

Overlap Theorem. The union and the intersection of two open (resp. closed) overlapping intervals are open (resp. closed) intervals.

Crossing Theorem. If A,B,C are noncollinear, and if ωADB and ωAEC, then there is a point F such that ωCFD and ωBFE.

4. *Connectivity in ordered planes.*

We now mention some theorems on geometric connectivity of sets. These theorems can be found in Valentine [14], where they are applied to Euclidean spaces, but they have never, as far as we know, been applied to arbitrary ordered planes. Again, we omit the proofs for most of these theorems.

A set S in an ordered plane is *intrinsically polygonally connected* if and only if, for every pair of points $A,B \in S$, there is a finite sequence $\{C_0,C_1,C_2,\ldots,C_n\}$ such that $C_0 = A$, $C_n = B$, and the closed intervals $[C_iC_{i+1}]$ are all contained in S. If it turns out that each interval is order isomorphic to the real unit interval [0,1], then the set is said to be *extrinsically polygonally connected.* This last concept is not important to our point of view, so "polygonally connected" will be intrinsic unless noted otherwise.

Discrimination Theorem. The rational plane is intrinsically but not extrinsically polygonally connected.

A set S is *starshaped with respect to a point* P if and only if, for any $Q \in S$, the closed interval [PQ] is contained in S. A set S is *starshaped* if and only if there is some point P of S such that the set is starshaped with respect to P. The *kernel* of a starshaped set is the set of all points P such that the set is starshaped with respect to P.

Inclusion Theorem I. If a set is starshaped, then it is intrinsically polygonally connected, but not conversely.

Inclusion Theorem II. If a set is convex, then it is starshaped, but not conversely.

Intersection Theorem. The intersection of two convex sets is convex.

Union Theorem. If the intersection of two convex sets is nonempty, then their union is starshaped and its kernel contains the intersection, sometimes properly so.

Kernel Theorem. The kernel of a starshaped set S is convex.

Outline of proof. Let A and C be points of the kernel, and let B be a point such that ωABC. It suffices to show that, for any point $P \in S$, the closed interval [BP] is contained in S. If P is on line AC, then [BP] is contained either in [AP] or in [CP], both of which are contained in S, so we are done. If P is not on line AC, let Q be any point such that ωBQP. From the Transversal Theorem, there is a point R such that ωAQR and ωCRP. Now R $\in S$ since C is in the kernel, and Q $\in S$ since A is in the kernel, so we are done.

5. *Topological concepts in ordered planes.*

We will in this section prove two useful theorems. With the first, the Extraction Theorem, we can construct new ordered planes from a given one; the other, the Connectivity Theorem, shows that topological connectivity implies a geometrical connectivity. The proofs will make use of some concepts from Borsuk and Szmielew [3] which are summarized below. All page numbers in this section are for reference [3].

A set X is a *(Hausdorff) topological space with base* β *of neighborhoods* if and only if β is a family of subsets of X satisfying

(i) if x and y are distinct points of X, then there are disjoint sets $U,V \in \beta$ such that $x \in U$ and $y \in V$;

(ii) if $U,V \in \beta$ and $x \in U \cap V$, then there is at least one set $W \in \beta$ such that $x \in W$ and W is contained in $U \cap V$.

In order to use Borsuk and Szmielew, it is necessary to establish their axioms as theorems for ordered planes. With a copy of Forder [5], it is not too hard to verify I1-I4 on page 21 concerning incidence, O1-O8 on page 26 concerning order, and O9 on page 42 is the Pasch axiom. On pages 43-46, they establish *Hilbert's Separation Theorem*: any line k divides the plane into two nonempty disjoint convex sets S and T (i.e., k,S,T are pairwise disjoint and their union is Σ); and if A $\in S$ and C $\in T$, then there is a point B $\in k$ such that ωABC.

In some sense, open half planes can be defined as sets S and T while closed half planes are $S \cup k$ and $T \cup k$. Moreover, with the Hilbert theorem we can also establish for the line that a point separates it into two nonempty disjoint convex sets called *open rays* while *closed rays* include the dividing point. Also, on the line we have the property that if A belongs to one open ray, C to the other, and B is the dividing point, then ωABC.

One page 32 it is shown that the open intervals of a line form a base for a topology on that line. On page 59, *open triangular regions* (ABC) are defined for noncollinear points A,B,C as the intersection of open half planes: one being the half plane determined by line BC and containing A, another the half plane determined by line CA and containing B, and the third the half plane determined by line AB and containing C. On pages 64-65, it is shown that these open triangular regions form a base for a topology of the plane. Therefore these regions are the neighborhoods with which we can define the concepts of interior points and boundary points of sets, which in turn serve to define open and closed sets, much as is done in the Euclidean plane with circular neighborhoods.

We are now ready to state and prove our two theorems.

Extraction Theorem. Let U be any nonempty open convex set of an ordered plane (Σ,ω). If ω^* denotes the restriction of ω to U, then (U,ω^*) is also an ordered plane.

Proof. Axioms O_1 and O_2 hold for ω^* since they hold for ω. For O_3, it is helpful to know the equivalent formulation for open triangular regions used in [3] (Theorem 74 on page 60): an open triangular region (ABC) is the set of all points P for which there is a point Q such that ωAPQ and ωBQC. Since U is open, the point B of O_3 has a neighborhood contained in U, say (XYZ), so there is a point D such that ωXBD and ωYDZ. If AB is also XD, then, from Hilbert's separation of a line, either A and D are on the same side of B or A and X are on the same side of B. In the former case, let C be such that ωBCX. Then it can be shown that ωABC and ωXCD, and so C is in (XYZ) and therefore in U. In the latter case, we let C be such that ωBCD and show that ωABC and ωXCD, and again $C \in U$. If AB is not CD, applying the Pasch axiom to triangle XDY, we obtain either (i) $Y \in AB$, or (ii) $AB \cap XY = E$ and ωXEY, or (iii) $AB \cap YD = E$ and ωYED; and similar statements can be made for triangle XDZ. If (i) is true, then we can find a point C just as we did before but with Y or Z playing the role of X. Of the remaining possibilities, we will concern ourselves with only one typical case, the proofs of all being similar. We assume that $AB \cap XY = E$ and ωXEY, and $AB \cap XZ = F$ and ωXFZ; we assume furthermore that in the case under consideration A and E are on the same side of B. We take for C any point such that ωBCF; then ωABC and, applying the Transversal Theorem to triangle FBZ and then to triangle BZD, we find a point G such that ωXCG and ωDGZ. Thus ωYGZ, so C belongs to (XYZ) and therefore to U.

For O_4, we use a neighborhood (XYZ) of some point $P \in U$. By Fano's Theorem, points K,L,M such that ωXKY, ωYLZ, ωZMX are noncollinear, as are points Q,R,S such that ωKQL, ωLRM, ωMSK. Using the intersection of half planes definition of a neighborhood, it is not too hard to show that the points Q,R,S are in the neighborhood since they cannot be on the boundary. So they are in U, which therefore contains three noncollinear points.

To show O_5, we simply note that line AB of the given ordered plane is the same as line CD, so the line AB of the new structure, which is $AB \cap U$, is the same as line CD of the new structure. To show O_6, the only point to be concerned about is E, which, by the convexity of U, belongs to U. □

Now we come to a theorem of Ahlfors [1] which depends mostly on the convexity of neighborhoods, so it is not surprising that it turns out to be true in an arbitrary ordered plane as well as in the complex plane. But first we need to define *topologically connected open sets*. These are the open sets which are not the union of two nonempty disjoint open sets.

Connectivity Theorem. If an open set S is topologically connected, then it is polygonally connected.

Proof. If S is not polygonally connected, then there are two points $A,B \in S$ which are not joined by closed intervals of S. Let U be the set of all points which are polygonally connected to A, and V the set of all points not polygonally connected to A. It now suffices to show that both U and V are open. Let $P \in U$. Since S is open, there is a triangular neighborhood (XYZ) of P contained in S. By convexity, for any point Q of this neighborhood, the interval [PQ] is in S, and so Q is polygonally connected to A. Thus U is open.

Let $P \in V$, and let neighborhood (XYZ) of P be contained in S. If Q is in this neighborhood, and if Q is connected to A, then A can be connected to P, a contradiction. Thus all points of (XYZ) are in V, and so V is open. □

6. *Complete ordered planes.*

The Converse of the Connectivity Theorem is true for some ordered planes (e.g., the complex plane [1]) but not for all. For example, the rational plane is the union of two nonempty disjoint open sets,

$$\{(x,y) \mid x < \sqrt{2}\} \quad \text{and} \quad \{(x,y) \mid x > \sqrt{2}\}.$$

It follows that the rational plane, though polygonally connected (Discrimination Theorem), is not topologically connected. So the converse will hold only for those ordered planes which have some kind of completeness property. To identify these, we will need

Dedekind's Axiom. If all the points of a line k are divided into two disjoint sets α and β such that

(i) each set has at least two points;

(ii) if $A \in \alpha$ and $B_1, B_2 \in \beta$, then not $\omega B_1 A B_2$;

(iii) if $B \in \beta$ and $A_1, A_2 \in \alpha$, then not $\omega A_1 B A_2$; then there is a point C such that for any $A \neq C$ in α and for any $B \neq C$ in β we have ωACB.

A *complete ordered plane* is an ordered plane (Σ,ω) for which Dedekind's Axiom also holds.

In a complete ordered plane, the converse of the Connectivity Theorem is also true. Here we merely sketch a proof of this result.

We assume that an open set S of a complete ordered plane is not topologically connected but is polygonally connected, and find a contradiction. By our first assumption, there are two nonempty disjoint open sets T and Φ whose union is S. Let $U \in T$ and $V \in \Phi$. By our second assumption, there is a finite sequence

$$W_0 = U, W_1, W_2, \ldots, W_n = V$$

with intervals $[W_i W_{i+1}]$ in S. Let j be the index such that $W_i \in T$ for all $i \leq j$ and $W_{j+1} \in \Phi$. Let α be the set of all points P of line $W_j W_{j+1}$ such that $\omega P W_j W_{j+1}$, or $P = W_j$, or both $\omega W_j P W_{j+1}$ and $[W_j P] \subseteq T$; and let β contain all the other points of the line. We will verify (ii) and (iii) of Dedekind's Axiom; (i) is much easier and is left to the reader.

First we note that if $\omega A W_j W_{j+1}$, then by Hilbert's Separation Theorem of the line by W_j we conclude that A is not on the same ray as B_1 and B_2, and therefore $\omega B_1 A B_2$ is impossible. Similarly, if $\omega W_j W_{j+1} B$, we can use W_{j+1} to conclude that B is not on the same ray as A_1 and A_2, and therefore $\omega A_1 B A_2$ is impossible. Similar arguments show that we cannot have either $W_j = A$ or $W_{j+1} = B$. Thus we have both $\omega W_j A W_{j+1}$ and $\omega W_j B W_{j+1}$.

It is obvious that one of $\omega B_1 W_j A$ or $\omega W_j B_1 A$ or $\omega W_j A B_1$ holds, but only the last is possible. In the same way, we conclude that $\omega W_j A B_2$ is possible. Hence, from the lemmas developed to prove the Infinitude Theorem (see Forder [5]), we do get $\omega A B_1 B_2$ or $\omega A B_2 B_1$, and (ii) holds.

If $\omega A_1 B A_2$ and either $\omega A_1 W_j B$ or $\omega W_j A_1 B$, then we get $\omega W_j B A_2$, a contradiction. Hence $\omega W_j B A_1$, which is also a contradiction, and this establishes (iii).

So there is a point C which satisfies the conclusion of Dedekind's Axiom, and we will have the desired contradiction when we have shown that $C \notin \alpha$ and $C \notin \beta$. If $C \in \alpha$, then with the approach used in proving the Extraction Theorem we can find a point $D \in T$ such that $\omega W_j C D$ and for which $[W_j D] \subseteq T$; hence $D \in \alpha$. Now ωBCD, $\omega W_j C D$, and $\omega W_j C B$ must hold simultaneously. But this is contrary to one of the aforementioned lemmas. In a similar way, we can show that $C \notin \beta$.

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* * *

THE PUZZLE CORNER

Puzzle No. 35: Deletion (7) (PRIME, NEXT)
Metathesis (6) (NEXT, FINE)

See Thessalonians, twenty-one, of five.
This Bible PRIME (I FINE you will agree)
Applies to mathematicians now alive;
But NEXT no other folk, whoe'er they be.

Puzzle No. 36: Rebus (9)

D

This letter does not mar or else MY KEY
This page; it is five hundred that you see.

ALAN WAYNE, Holiday, Florida

* * *

PALINDROMIC EQUIVALENTS OF THREE-DIGIT DECIMAL PALINDROMES

CHARLES W. TRIGG

There are thirty-two three-digit palindromes in the decimal system whose equivalents in smaller bases b, $2 \le b \le 9$, are also palindromic. Six of these have palindromic equivalents in two other bases, and two have palindromic equivalents in three other bases. These matched palindromes are listed below. (Numbers throughout are in base ten unless otherwise indicated.)

$111 = 303_{\text{six}}$

$121 = 11111_{\text{three}} = 232_{\text{seven}} = 171_{\text{eight}}$

$141 = 353_{\text{six}}$

$151 = 12121_{\text{three}}$

$171 = 333_{\text{seven}}$

$191 = 515_{\text{six}} = 232_{\text{nine}}$

$212 = 21212_{\text{three}}$

$242 = 22222_{\text{three}} = 464_{\text{seven}}$

$252 = 2002_{\text{five}}$

$282 = 2112_{\text{five}} = 343_{\text{nine}}$

$292 = 565_{\text{seven}} = 444_{\text{eight}}$

$313 = 100111001_{\text{two}}$

$333 = 515_{\text{eight}}$

$343 = 1331_{\text{six}}$

$373 = 11311_{\text{four}} = 565_{\text{eight}} = 454_{\text{nine}}$

$393 = 12021_{\text{four}}$

$414 = 636_{\text{eight}}$

$434 = 2002_{\text{six}}$

$464 = 565_{\text{nine}}$

$484 = 122221_{\text{three}}$

$555 = 676_{\text{nine}}$

$585 = 1001001001_{\text{two}} = 1111_{\text{eight}}$

$626 = 10001_{\text{five}}$

$646 = 787_{\text{nine}}$

$656 = 220022_{\text{three}} = 808_{\text{nine}}$

$666 = 22122_{\text{four}}$

$676 = 10201_{\text{five}}$

$717 = 1011001101_{\text{two}}$

$757 = 1001001_{\text{three}}$

$777 = 3333_{\text{six}}$

$868 = 4004_{\text{six}}$

$939 = 32223_{\text{four}}$

Five of the decimal palindromes and six of the equivalent palindromes are repdigits. Indeed, both 777 and its equivalent are repdigits.

Both 212 and its equivalent undulate smoothly with the same digits. All digits of 242, 868, and their equivalents are even. All digits of 171, 333, 777, and their equivalents are odd. The equivalents of 343, 434, 484, 626, 868, and 939 are plateau numbers.

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THE OLYMPIAD CORNER: 44

M.S. KLAMKIN

I give three new problem sets this month. The first is Practice Set 18, for which solutions will appear here next month. The second, which I am able to present through the courtesy of Jordan B. Tabov, and for which I solicit elegant solutions from readers, consists of the problems set at the Bulgarian Winter Competition held in Varna on January 17, 1983. The third, for which I also solicit solutions, consists of the problems set at the 1983 British Mathematical Olympiad. Finally, I present solutions to last month's Practice Set 17.

PRACTICE SET 18

18-1. If z_0 is any zero of the polynomial

$$f(z) = a_0z^n + a_1z^{n-1} + \ldots + a_n,$$

where $a_0 \geq a_1 \geq \ldots \geq a_n \geq 0$, prove that $|z_0| \leq 1$.

18-2. A,B,C,D,E are five coplanar points such that no two of the lines joining these points in pairs are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to the lines which join the remaining points in pairs. Determine the maximum number of points of intersection of these perpendiculars.

18-3. Show how to construct a triangle having its vertices on three given skew lines so that the centroid of the triangle coincides with a given point.

*

BULGARIAN WINTER COMPETITION

January 17, 1983

1. Prove that a regular hexagon with edge of length 2 can be covered by six circular disks of unit radius but cannot be covered by five such disks.

2. A right triangular prism OAB-O'A'B' is given whose base is an isosceles right triangle AOB with right angle at O. It is known that there exist points M and N on segment A'B, and points P and Q on segment OA', such that MNPQ is a regular tetrahedron. Determine the volume of the prism, given that the distance between the midpoints of MN and PQ is d.

3. Determine all values of the real parameter p for which the system of equations

$$\begin{aligned} x + y + z &= 2 \\ yz + zx + xy &= 1 \\ xyz &= p \end{aligned}$$

has a real solution.

*

BRITISH MATHEMATICAL OLYMPIAD

10 March 1983 — Time: 3½ hours

1. In the triangle ABC with circumcentre O, AB = AC, D is the midpoint of AB, and E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

2. The *Fibonacci sequence* $\{f_n\}$ is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (n > 2).$$

Prove that there are unique integers a, b, m such that $0 < a < m$, $0 < b < m$, and $f_n - anb^n$ is divisible by m for all positive integers n.

3. The real numbers $x_1, x_2, x_3, \ldots$ are defined by

$$x_1 = a \neq -1 \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \text{ for all } n \geq 1.$$

S_n is the sum and P_n is the product of the first n terms of the sequence $y_1, y_2, y_3, \ldots$, where

$$y_n = \frac{1}{1 + x_n}.$$

Prove that $aS_n + P_n = 1$ for all n.

4. The two cylindrical surfaces

$$x^2 + z^2 = a^2, \quad z > 0, \quad |y| \leq a$$

and

$$y^2 + z^2 = a^2, \quad z > 0, \quad |x| \leq a$$

intersect, and with the plane $z = 0$ enclose a dome-like shape which is here called a *cupola*. The cupola is placed on top of a vertical tower of height h whose horizontal cross-section is a square of side $2a$. Find the shortest distance from the highest point of the cupola to a corner of the base of the tower, over the surface of the cupola and tower.

5. If 10 points are within a circle of diameter 5 inches, prove that the distance between some 2 of the points is less than 2 inches.

6. Consider the equation

$$\sqrt{2p+1-x^2} + \sqrt{3x+p+4} = \sqrt{x^2+9x+3p+9}, \tag{1}$$

in which x,p are real numbers and the square roots are to be *real* and *nonnegative*. Show that if (1) holds, then

$$(x^2 + x - p)(x^2 + 8x + 2p + 9) = 0.$$

Hence find the set of real numbers p for which (1) is satisfied by exactly one real number x.

*

PRACTICE SET 17

17-1. [1983: 72] The numbers 1, 2, 3, ..., n^2 are partitioned into n groups of n numbers. If s_j denotes the sum of the numbers in the jth group, determine the maximum and minimum values of $S \equiv s_1 s_2 \ldots s_n$.

Rider. Find the number of distinct partitions which yield the maximum value of S.

Solution.

(a) *Maximum value of S.* By the A.M.-G.M. inequality, we have

$$\frac{1}{n}(s_1 + s_2 + \ldots + s_n) \geq \sqrt[n]{s_1 s_2 \ldots s_n},$$

with equality if and only if $s_1 = s_2 = \ldots = s_n$. Hence

$$S \leq \left\{\sum_{i=1}^{n} \frac{s_i}{n}\right\}^n = \left\{\sum_{j=1}^{n^2} \frac{j}{n}\right\}^n = \left\{\frac{n(n^2+1)}{2}\right\}^n.$$

Thus we will have

$$\max S = \left\{\frac{n(n^2+1)}{2}\right\}^n$$

provided there exists a partition such that $s_1 = s_2 = \ldots = s_n$. Such a partition exists for every n.

For even n, we can write the numbers in increasing order alternately from left to right and right to left (the first array illustrates the case $n = 4$). The partition is then given by the n columns, each of which has a sum of $n(n^2+1)/2$.

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

For odd n, we can always construct a magic square of order n by following the rule of De La Loubère [1] (the second array illustrates the case $n = 5$). The partition is then given either by the five rows or by the five columns, each of which has a sum equal to the magic constant $n(n^2+1)/2$. (We can also use a magic square for even n, but the rule for constructing it, which is described in [1], is more complicated than for odd n.)

(b) *Minimum value of S.* We may assume without loss of generality that $s_1 \le s_2 \le \ldots \le s_n$. Then $s_1 \ge 1 + 2 + \ldots + n = n(n+1)/2$. If $s_1 > n(n+1)/2$, then some number $a > n$ belongs to s_1 while some number $b \le n$ belongs to s_t for some $t > 1$. If $d = a-b > 0$, then

$$(s_1-d)(s_t+d) = s_1s_t - d(s_t-s_1) - d^2 < s_1s_t,$$

which shows that S can be reduced by interchanging a and b. It follows that for minimal S we must have

$$s_1 = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

A similar proof shows that for minimal S we must have

$$\begin{aligned}
s_2 &= (n+1) + (n+2) + \ldots + (n+n) = s_1 + n^2,\\
s_3 &= (2n+1) + (2n+2) + \ldots + (2n+n) = s_1 + 2n^2,\\
&\vdots\\
s_n &= \{(n-1)n+1\} + \{(n-1)n+2\} + \ldots + \{(n-1)n+n\} = s_1 + (n-1)n^2.
\end{aligned}$$

Finally,

$$\min S = \prod_{k=0}^{n-1} (s_1 + kn^2) = \prod_{k=0}^{n-1} \left(\frac{n(n+1)}{2} + kn^2\right).$$

REFERENCE

1. W.W. Rouse Ball and H.S.M. Coxeter, *Mathematical Recreations and Essays*, Twelfth Edition, University of Toronto Press, 1974, pp. 193-200.

17-2. [1983: 72] ABCD is a skew quadrilateral (i.e., with sides not all coplanar) such that

$$\angle ABC = \angle BCD = \angle CDA = 90^\circ.$$

Prove that $\angle DAB$ is acute.

Solution.

Consider the two triangles ABD and CBD formed by drawing diagonal BD. Since the sum of any two face angles of a trihedral angle is greater than the third face angle, we have

$$\angle CDB + \angle ADB > \angle CDA = 90^\circ$$

and

$$\angle CBD + \angle ABD > \angle ABC = 90^\circ.$$

Hence

$$\begin{aligned}\angle DAB &= 360^\circ - \angle ADB - \angle ABD - \angle CDB - \angle CBD - \angle BCD \\ &< 360^\circ - 90^\circ - 90^\circ - 90^\circ = 90^\circ,\end{aligned}$$

and $\angle DAB$ is acute.

17-3. [1983: 72] The equation $(x-a_1)(x-a_2)\dots(x-a_n) = 1$, where the a_i are all real, has n real roots r_i. Find the minimum number of real roots of the equation

$$(x-r_1)(x-r_2)\dots(x-r_n) = -1.$$

Solution.

We must have

$$(x-a_1)(x-a_2)\dots(x-a_n) - 1 = (x-r_1)(x-r_2)\dots(x-r_n);$$

hence the roots of $(x-r_1)(x-r_2)\dots(x-r_n) = -1$ are $a_1, a_2, \dots, a_n$, and their number is exactly n.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

MATHEMATICAL CLERIHEWS

G. H. Hardy
Wasn't tardy.
His work was good
(With Littlewood.)

Hermann Weyl
Wrote with style,
Not idle chatter,
In *Space, Time, Matter*.

ALAN WAYNE, Holiday, Florida

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

831. *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

The adjoined decimal alphametic must represent the occasional feelings of our harried French editor. Fortunately, his surname ensures his salvation, and its primeness ensures his uniqueness.

```
 ASSEZ
 ASSEZ
 ASSEZ
  SAYS
   LEO
 -----
 SAUVE
```

832. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Let S be a subset of an $m \times n$ rectangular array of points, with $m,n \ge 2$. A *circuit in S* is a simple (i.e., nonself-intersecting) polygonal closed path whose vertices form a subset of S and whose edges are parallel to the sides of the array.

Prove that a circuit in S always exists for any subset S with $|S| \ge m+n$, and show that this bound is best possible.

833. *Proposed by Charles W. Trigg, San Diego, California.*

(a) What is the largest integer, a permutation of the nine nonzero digits, that is divisible by 99?

(b) What is the smallest such number divisible by 99?

(c) If the nine nonzero digits are arranged at random, what is the probability that the integer formed will be divisible by 99?

(d) Answer (a), (b), and (c) as applied to integers formed from the ten distinct digits (initial zeros excluded).

834. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

For fixed positive integer n and arbitrary θ, simplify the product

$$\sin\theta\sin\left(\theta+\frac{2\pi}{n}\right)\sin\left(\theta+\frac{4\pi}{n}\right)\ldots\sin\left(\theta+\frac{2(n-1)\pi}{n}\right).$$

835. *Proposed by Jack Garfunkel, Flushing, New York; and George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with sides a,b,c, and let R_m be the circumradius of the triangle formed by using as sides the medians of triangle ABC. Prove that

$$R_m \geq \frac{a^2 + b^2 + c^2}{2(a + b + c)}.$$

836. *Proposed by Vedula N. Murty, Pennsylvania State University, The Capitol Campus.*

(a) If A,B,C are the angles of a triangle, prove that

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C,$$

with equality if and only if the triangle is equilateral.

(b) Deduce from (a) Bottema's triangle inequality [1982: 296]:

$$(1 + \cos 2A)(1 + \cos 2B)(1 + \cos 2C) + \cos 2A \cos 2B \cos 2C \geq 0.$$

837. *Proposed by R.B. Killgrove, University of South Carolina at Aiken.*

Let G be a group with normal subgroups H and K. Prove or disprove:

(a) if H and K are isomorphic, then G/H and G/K are also isomorphic;

(b) if G/H and G/K are isomorphic, then H and K are also isomorphic.

838. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

For a given regular tetrahedron of edge length 2, there is a point M such that the distance between M and each of the four vertices is an integer. Prove that M must coincide with one of the vertices of the tetrahedron.

839. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Prove that the Diophantine equation

$$x^n + y^n = z^{n+1}$$

has infinitely many solutions (x,y,z) for every natural number n.

840. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find the remainder when 72! is divided by 79.

* * *

THE PUZZLE CORNER

	0	1	2	3	4	5	6	7	8	9
Answer to Puzzle to No. 31 [1983: 71]:	K	U	S	T	E	M	A	B	R	I
Answer to Puzzle to No. 32 [1983: 71]:	*	S	E	T	Y	*	*	U	R	M
Answer to Puzzle to No. 33 [1983: 71]:	C	P	I	E	U	M	L	A	R	S
Answer to Puzzle to No. 34 [1983: 71]:	A	F	N	T	D	*	*	R	*	O

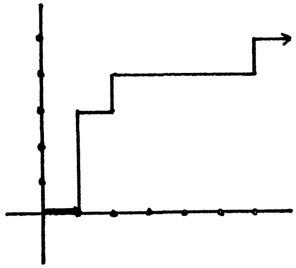
In Puzzle No. 31, note that RUBIK^3 = MEBUESBUITTIKKK, which starts with ME.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

408. [1979: 16, 277, 294] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A *zigzag* is an infinite connected path in a Cartesian plane formed by starting at the origin and moving successively one unit right or up (see figure). Prove or disprove that for every zigzag and for every positive integer k, there exist (at least) k collinear lattice points on the zigzag.



(This problem was given to me by a classmate at City College of New York in 1971-72. Its origin is unknown to me.)

II. *Comment by George Tsintsifas, Thessaloniki, Greece.*

It has apparently so far escaped notice that this problem is the same as Problem 5811 proposed by T.C. Brown in the September 1971 issue of the *American Mathematical Monthly*. A solution by P.L. Montgomery was published in the December 1972 *Monthly*.

Editor's comment.

It is now clear why this problem was "in the air" in 1971-72, when our proposer heard about it from a classmate.

* * *

711. [1982: 47; 1983: 53] *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

Find all the solutions of the following alphametic (in which the last word represents the sum of the preceding words), which I have worked on from time to time but never carried to completion:

A ROSE IS A ROSE IS A ROSE SO THE WARS OF THE ROSES AROSE.

III. *Comment by J.A.H. Hunter, Toronto, Ontario.*

The 9-addend alphametic mentioned by Stanley Rabinowitz [1983: 54] was hardly a "monster" in comparison with a 41-addend real monster that appeared in the *Journal of Recreational Mathematics*, Vol. 5, No. 4, 1972, p. 289. This monster was proposed by Anton Pavlis, of Guelph, Ontario. It is shown on the following page, along with its unique solution, which was arrived at and confirmed by computer.

SO	31
MANY	2764
MORE	2180
MEN	206
SEEM	3002
TO	91
SAY	374
THAT	9579
THEY	9504
MAY	274
SOON	3116
TRY	984
TO	91
STAY	3974
AT	79
HOME	5120
SO	31
AS	73
TO	91
SEE	300
OR	18
HEAR	5078
THE	950
SAME	3720
ONE	160
MAN	276
TRY	984
TO	91
MEET	2009
THE	950
TEAM	9072
ON	16
THE	950
MOON	2116
AS	73
HE	50
HAS	573
AT	79
THE	950
OTHER	19508
TEN	906
TESTS	90393

Editor's comment.

On the same page 289 of the *Journal of Recreational Mathematics* where the monster appeared was the following alphametic:

$$\begin{array}{r} \text{ODD} \\ \text{ODD} \\ \underline{\text{OLD}} . \\ \text{LEO} \end{array}$$

Compare with Crux 831 in this issue (by, as it happens, the same proposer as the present problem). Unused to being the butt of jokes, this odd, odd, old (61) editor vows to get even (62) before long.

* * *

727. [1982: 78] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let t_b and t_c be the symmedians issued from vertices B and C of triangle ABC and terminating in the opposite sides b and c, respectively. Prove that $t_b = t_c$ if and only if $b = c$.

Solutions were received from KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; DIMITRIS VATHIS, Chalcis, Greece; and the proposer. Comments were received from M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire.

Editor's comment.

As can be seen in references [1]-[3] given below, all of which were sent in by readers, this problem is very well known. The solutions received were all more or less equivalent to the already published ones. Rabinowitz also gave reference [4] for a similar result about exsymmedians (a triangle is isosceles if and only if it has two equal exsymmedians).

REFERENCES

1. L.M. Kelly, Solution of Problem E 613 (Equal Symmedians), *American Mathematical Monthly*, 51 (December 1944) 590-591.

2. V. Thébault, "Recreational Geometry—The Triangle", *Scripta Mathematica*, 22 (1956) 102.

3. Problem 637 (proposed by Stanley Rabinowitz), solution by Leon Bankoff, *Mathematics Magazine*, 40 (May 1967) 165; comment by the proposer, *ibid.*, 41 (January 1968) 48-49.

4. Problem 213, solution by proposer Gregory Wulczyn, *Pi Mu Epsilon Jouranl*, Vol. 5, No. 2 (Spring 1970), p. 88.

* * *

728. [1982: 78] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let $E(P,Q,R)$ denote the ellipse with foci P and Q which passes through R. If A,B,C are distinct points in the plane, prove that no two of $E(B,C,A)$, $E(C,A,B)$, and $E(A,B,C)$ can be tangent.

Solution by Gali Salvatore, Perkins, Québec.

We will show that the conclusion of the theorem is true if and only if the three distinct points A,B,C are not collinear. For if they are collinear with, say, B between A and C, and if we let $E(C,A,B)$ be the degenerate ellipse consisting of the segment AC, then clearly $E(C,A,B)$ and $E(A,B,C)$ are tangent at C, while $E(C,A,B)$ and $E(B,C,A)$ are tangent at A.

We now assume that A,B,C are not collinear. By symmetry, it suffices to show that $E(B,C,A)$ and $E(C,A,B)$ are not tangent. Suppose they are tangent at T. Then A,B,C are all on the same side of the common tangent at T. None of A,B,C coincides with T, for otherwise at least one ellipse is degenerate and A,B,C are collinear. From a known property of the ellipse, the reflection of TC in the common normal at T coincides with TA as well as with TB. Hence T,A,B are collinear. Since B is a focus of one ellipse, and T and A are distinct points on that ellipse, B separates T and A. Also, A is a focus of the other ellipse, and T and B are distinct points on it, so A separates T and B. The contradiction shows that the two ellipses are not tangent.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; R.C. LYNESS, Southwold, Suffolk, England; LEROY F. MEYERS, The Ohio State University; DAN PEDOE, University of Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

All but two of the other solvers tacitly assumed that A,B,C were noncollinear, a fact that was not guaranteed by the proposal. The two exceptional solvers decided that the conclusion of the theorem holds even when A,B,C are collinear because, apparently, they did not consider that degenerate ellipses were bona-fide ellipses. But it never hurts to take the semantically larger view in mathematics, as did our featured solver.

* * *

729. [1982: 78] *Proposed jointly by Dick Katz and Dan Sokolowsky, California State University at Los Angeles.*

Given a unit square, let K be the area of a triangle which covers the square. Prove that $K \geq 2$.

Solutions were received from ELWYN ADAMS, Gainesville, Florida; KENT BOKLAN, student, Massachusetts Institute of Technology; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; NICK MARTIN, student, Indiana University at Bloomington; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (partial solution); ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; and the proposer. Comments were received from JOEL BRENNER, Palo Alto, California; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; JOE KONHAUSER, Macalester College, Saint Paul, Minnesota; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire.

Editor's comment.

As several readers pointed out in their comments, the problem is not new. It has already appeared in the *Monthly* [3] (*everything* has appeared before in the *Monthly*!), where an editor's comment noted that in the problem the word "square" can be replaced by the word "parallelogram", for the one can be carried into the other by an affine transformation, which multiplies all areas by a constant factor. For more general results, see [1], [2], and [4].

REFERENCES

1. H.G. Eggleston, *Problems in Euclidean Space*, Pergamon Press, New York, 1957, p. 156, Theorems 9.4 and 9.5.

2. Curtis M. Fulton and Sherman K. Stein, "Parallelograms Inscribed in Convex Curves", *American Mathematical Monthly*, 67 (1960) 257-258.

3. Problem E 1425 (proposed by D.J. Newman, solution by the proposer), *American Mathematical Monthly*, 68 (1961) 180-181.

4. Problem 161 (proposed by Paul Schillo), *Pi Mu Epsilon Journal*, 4 (Spring 1967) 259.

* * *

730. [1982: 78] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Prove that $D = 0$ for all real θ if $D = |a_{ij}|$ is a determinant with

$$a_{ij} = \cos(i+j)\theta, \quad i,j = 0,1,2,\dots,n; \quad n \geq 2.$$

Solution by M.S. Klamkin, University of Alberta.

More generally, let D be be the determinant with the first three rows as given, the remaining rows being arbitrary. (Since $n \geq 2$, there are at least three rows.) We show that $D = 0$ for all complex θ. If the first row is added to the third, the new third row is

$$\cos(j+2)\theta + \cos j\theta = 2\cos\theta\cos(j+1)\theta, \qquad j = 0,1,\ldots,n.$$

Thus the new third row is proportional to the second row, and so $D = 0$.

In the same way, we can show still more generally that $D = 0$ if

$$a_{ij} = \cos(ui+vj+w)\theta,$$

or $\qquad\qquad i = 0,1,2;\ j = 0,1,2,\ldots,n,$

$$a_{ij} = \sin(ui+vj+w)\theta.$$

Also solved by CURTIS COOPER, Central Missouri State University at Warrensburg; J.T. GROENMAN, Arnhem, The Netherlands; NICK MARTIN, student, Indiana University at Bloomington; LEROY F. MEYERS, The Ohio State University; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack New Hampshire (two solutions); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

* * *

731. [1982: 106] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Pity poor Stan. He built a better rat trap and, although he used only prime TRAPS, the world did not beat a path to his door: all he got was better rats. So, regretfully, we continue to urge:

$$\begin{array}{r} \mathrm{TRAP} \\ \mathrm{RATS} \\ \mathrm{STAN} \\ \mathrm{IN} \\ \mathrm{RAT} \\ \hline \mathrm{TRAPS} \end{array}.$$

I. *Solution by Judith Relic, Cleveland, Ohio.*

Let a,b,c be the carries into the second, third, and fourth columns, respectively, counting from the right. Immediately, we see that $1 \le a \le 3$, $0 \le b \le 4$, $1 \le c \le 3$, $\mathrm{T} \in \{1,2\}$, and $\mathrm{S} \in \{1,3,7,9\}$ since TRAPS is prime.

If $\mathrm{T} = 2$, then $20 = c+2+\mathrm{S} \le \mathrm{S}+5$ and there is no value for S. Hence $\underline{\mathrm{T} = 1}$. Now $c+1+\mathrm{S} = 10$, so $6 \le \mathrm{S} \le 8$, from which $\underline{\mathrm{S} = 7}$ and $\underline{c = 2}$. From $\mathrm{P}+2\mathrm{N}+1 = 10a$, we conclude that P is odd and $a \in \{1,2\}$; and from $b+2\mathrm{R}+1 = 20$ that b is odd. Therefore $(b,\mathrm{R}) = (1,9)$ or $(3,8)$.

Suppose $(b,\mathrm{R}) = (3,8)$. If $a = 1$, then $3\mathrm{A}+\mathrm{I} = \mathrm{P}+28$, from which $\mathrm{A} = 9$, $\mathrm{P} = 5$, and TRAPS equals the nonprime 18957. If $a = 2$, then $3\mathrm{A}+\mathrm{I} = \mathrm{P}+27$ and $\mathrm{A} \le 6$, which leads to the impossible $\mathrm{I} \ge \mathrm{P}+9$. Therefore $\underline{(b,\mathrm{R}) = (1,9)}$.

From $a+3\mathrm{A}+1+\mathrm{I} = 10+\mathrm{P}$, it follows that $2 \le \mathrm{A} \le 5$. Since no combination of $\mathrm{A} \in \{3,4,5\}$ and $a \in \{1,2\}$ gives an acceptable value for P, except $(\mathrm{A},a) = (3,1)$ which yields the nonprime TRAPS = 19357, we conclude that $\underline{\mathrm{A} = 2}$.

If $a = 1$, duplications occur for $P \in \{3,5\}$, so $\underline{a = 2}$, and we quickly find $\underline{P = 3}$, $\underline{I = 4}$, and $\underline{N = 8}$. The unique reconstruction is

$$\begin{array}{r} 1923 \\ 9217 \\ 7128 \\ 48 \\ 921 \\ \hline 19237 \end{array},$$

with prime TRAPS.

II. *Comment by Robert S. Johnson, Montréal, Québec.*

[In the above unique solution], Stan's TRAPS form a prime collection, but no single TRAP is prime. Poor Stan might trap his better rats better with the following solution,

$$\begin{array}{r} 1459 \\ 4518 \\ 8150 \\ 20 \\ 451 \\ \hline 14598 \end{array},$$

in which each TRAP is prime.

Also solved by MEIR FEDER, Haifa, Israel; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; J.A. McCALLUM, Medicine Hat, Alberta; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

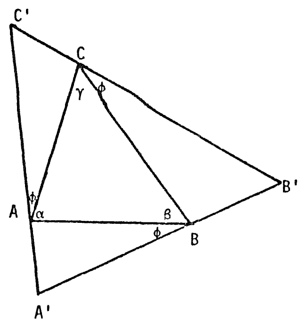
Readers will note from the proposal that the English language is not associative:

better (rat trap) ≠ (better rat) trap.

*　　　　*　　　　*

732. [1982: 106] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a fixed triangle ABC with angles α,β,γ and a variable circumscribed triangle A'B'C' determined by an angle $\phi \in [0,\pi)$, as shown in the figure. It is easy to show that triangles ABC and A'B'C' are directly similar.



(a) Find a formula for the ratio of similitude

$$\lambda = \lambda(\phi) = B'C'/BC.$$

(b) Find the maximal value λ_m of λ as ϕ varies in $[0,\pi)$, and show how to construct triangle A'B'C' when $\lambda = \lambda_m$.

(c) Prove that $\lambda_m \geq 2$, with equality just when triangle ABC is equilateral.

Solution by M.S. Klamkin, University of Alberta.

(a) The desired result follows very quickly from a known result of J. Neuberg,

$$\frac{K'}{K} = \left\{\frac{\sin(\omega+\phi)}{\sin\omega}\right\}^2, \tag{1}$$

where K and K' denote the areas of triangles ABC and A'B'C', respectively, and ω is the Brocard angle of triangle ABC. For

$$\lambda(\phi) = \frac{B'C'}{BC} = \sqrt{\frac{K'}{K}} = \frac{\sin(\omega+\phi)}{\sin\omega}. \tag{2}$$

(b) It is clear from (2) that the maximal value of $\lambda(\phi)$ occurs when

$$\omega + \phi = \frac{\pi}{2}, \tag{3}$$

and so $\lambda_m = \csc\omega$. To find the triangle A'B'C' corresponding to λ_m, we first construct one of the Brocard points Ω, Ω' of triangle ABC [1, p. 265]. As ϕ is defined in the figure, the one we need is Ω', which is uniquely defined by

$$\angle\Omega'CB = \angle\Omega'AC = \angle\Omega'BA = \omega.$$

Then, in accordance with (3), we simply draw $B'C' \perp \Omega'C$, $C'A' \perp \Omega'A$, and $A'B' \perp \Omega'B$.

(c) Since $\omega \leq 30^\circ$ [1, p. 270], we have $\lambda_m \geq \csc 30^\circ = 2$, with equality just when triangle ABC is equilateral. □

We found formula (1) in Casey's *Trigonometry*, where it is given incorrectly (with a square root instead of a square). So, for completeness, we present a derivation of (2).

If a,b,c are the sides and R the circumradius of triangle ABC, we have, by the law of sines,

$$B'C = \frac{a\sin(\phi+\beta)}{\sin\beta} = \frac{2R\sin\alpha\sin(\phi+\beta)}{\sin\beta}$$

and

$$CC' = \frac{b\sin\phi}{\sin\gamma} = \frac{2R\sin\beta\sin\phi}{\sin\gamma};$$

hence

$$B'C' = 2R\left\{\frac{\sin\alpha\sin(\phi+\beta)}{\sin\beta} + \frac{\sin\beta\sin\phi}{\sin\gamma}\right\}$$

and so

$$\frac{B'C'}{BC} = \frac{\sin(\phi+\beta)}{\sin\beta} + \frac{\sin\beta\sin\phi}{\sin\gamma\sin\alpha}$$

$$= \cos\phi + \sin\phi\left\{\frac{\cos\beta}{\sin\beta} + \frac{\sin\beta}{\sin\gamma\sin\alpha}\right\}.$$

Now [1, p. 267]

$$\frac{\cos\beta}{\sin\beta} + \frac{\sin\beta}{\sin\gamma\sin\alpha} = \frac{1+\cos\alpha\cos\beta\cos\gamma}{\sin\alpha\sin\beta\sin\gamma} = \cot\omega;$$

hence

$$\frac{B'C'}{BC} = \cos\phi + \sin\phi\cot\omega = \frac{\sin(\omega+\phi)}{\sin\omega}.$$

Also solved by JORDI DOU, Barcelona, Spain; HENRY E. FETTIS, Mountain View, California; JACK GARFUNKEL, Flushing, New York; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer.

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733. [1982: 106] *Proposed by Jack Garfunkel, Flushing, N.Y.*

A triangle has sides a,b,c, and the medians of this triangle are used as sides of a new triangle. If r_m is the inradius of this new triangle, prove or disprove that

$$r_m \le \frac{3abc}{4(a^2+b^2+c^2)},$$

with equality just when the original triangle is equilateral.

I. *Solution by George Tsintsifas, Thessaloniki, Greece.*

Let K and R be the area and circumradius, respectively, of the original triangle, ABC; and let K_m and s_m be the area and semiperimeter, respectively, of the *median triangle*, whose sides are the medians m_a, m_b, m_c of triangle ABC. It is well known [1] that $K_m = \frac{3}{4}K$. With this and the classical relations

$$abc = 4KR \qquad \text{and} \qquad K_m = r_m s_m,$$

the proposed inequality is easily shown to be equivalent to

$$2R(m_a + m_b + m_c) \ge a^2 + b^2 + c^2. \tag{1}$$

To establish (1), let G be the centroid of triangle ABC, and let g_a, g_b, g_c be its distances from sides a,b,c, respectively. It is known [2] that

$$a \cdot GA \geq bg_c + cg_b,$$

with equality if and only if $b = c$. Therefore, if h_a, h_b, h_c are the altitudes of triangle ABC, we have

$$2Rm_a = 3R\cdot GA \geq \frac{3R}{a}(bg_c + cg_b) = Rh_c\cdot\frac{b}{a} + Rh_b\cdot\frac{c}{a} = \frac{ab}{2}\cdot\frac{b}{a} + \frac{ca}{2}\cdot\frac{c}{a} = \frac{b^2+c^2}{2}, \qquad (2)$$

and (1) follows from (2) and two similar results, with equality if and only if $a = b = c$.

II. *Solution by M.S. Klamkin, University of Alberta and O.G. Ruehr, Michigan Technological University.*

[As shown in solution I, whose notation will be followed,] the proposed inequality is equivalent to (1), which we will establish by proving a stronger inequality.

From the polar moment of inertia inequality [3],

$$(x + y + z)^2R^2 \geq yza^2 + zxb^2 + xyc^2,$$

where x,y,z are arbitrary real numbers, we obtain

$$(m_a + m_b + m_c)^2R^2 \geq m_bm_ca^2 + m_cm_ab^2 + m_am_bc^2,$$

and (1) will follow immediately from

$$4(m_bm_ca^2 + m_cm_ab^2 + m_am_bc^2) \geq (a^2 + b^2 + c^2)^2. \qquad (3)$$

It is known (see [1982: 308] or [4]) that if

$$I(a,b,c,m_a,m_b,m_c) \geq 0$$

is a valid inequality relating the sides and medians of a triangle, then

$$I(m_a,m_b,m_c,\tfrac{3}{4}a,\tfrac{3}{4}b,\tfrac{3}{4}c) \geq 0$$

is also a valid inequality, and conversely. The median dual of (3) is

$$\frac{9}{4}(bcm_a^2 + cam_b^2 + abm_c^2) \geq (m_a^2 + m_b^2 + m_c^2)^2,$$

or, since $4m_a^2 = 2b^2+2c^2-a^2$, etc.,

$$bc(2b^2+2c^2-a^2) + ca(2c^2+2a^2-b^2) + ab(2a^2+2b^2-c^2) \geq (a^2 + b^2 + c^2)^2. \qquad (4)$$

Finally, a bit of algebra shows that (4) is equivalent to

$$\Sigma\{a^2(b-c)^2 - (b-c)^4\} = \Sigma(c+a-b)(a+b-c)(b-c)^2 \geq 0, \tag{5}$$

where the sums are cyclic over a,b,c. Now (5) is clearly true, since the summands are all nonnegative. So (3) and (1) are established, with equality if and only if $a = b = c$.

Also solved (by calculus) by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India. One incorrect solution was received.

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734. [1982: 106] *Proposed by H. Kestelman, University College, London, England.*

The first $n - 1$ columns of a real $n \times n$ matrix are given mutually orthogonal vectors of unit length. How can one choose the last column to ensure that the matrix is orthogonal and has determinant +1?

Solution by the proposer.

If the cofactors of the elements of the last column are $A_{1n}, A_{2n}, \ldots, A_{nn}$ (these depend only on the first $n-1$ columns), then these are the elements of the nth column of the required matrix. This is justified as follows.

The nth column has to be a unit vector orthogonal to those given; such a vector exists and is unique if the resulting matrix A is to have $\det A = +1$. Now $AA^T = I$ and $A \operatorname{adj} A = (\det A)I$ if A is orthogonal, and if $\det A = +1$ we need $A^T = \operatorname{adj} A$: hence the nth column of A and that of $(\operatorname{adj} A)^T$ are the same.

Also solved by W.C. IGIPS, Danbury, Connecticut; M.S. KLAMKIN, University of Alberta; and DAVID SINGMASTER, Polytechnic of the South Bank, London, England (three solutions).

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735. [1982: 107] *Proposed by S.C. Chan, Singapore.*

Solve the following problem, which is given without solution in Hall & Stevens, *A School Geometry*, Macmillan, London, 1944, page 310, Problem 50:

In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

Solution by Dan Pedoe, University of Minnesota.

More generally, given a conic Γ and three distinct points P,Q,R in the projective plane, we will find a triangle ABC inscribed in Γ such that the lines AB,BC,CA pass through P,Q,R, respectively.

For any point M on Γ, let

$$M_1 = MP \cap \Gamma, \qquad M_2 = M_1Q \cap \Gamma, \qquad \text{and} \qquad M' = M_2R \cap \Gamma.$$

By the projective theory of conics, the ranges {M} and {M'} are projective (see [1]). We will find the fixed points of these projective ranges, that is, the points M such that M' = M. If we take any three distinct points U,V,W on Γ and let

$$X = VW' \cap V'W \qquad \text{and} \qquad Y = WU' \cap W'U,$$

then the required fixed points are the intersections with Γ of the line XY.

The required triangle ABC is the one where A is either one of the fixed points (so there are in general two solutions), $B = A_1 = AP \cap \Gamma$, and $C = A_2 = BQ \cap \Gamma$, for then $CA = A_2A'$ passes through R. □

The proposed problem is the special case where Γ is a circle and one of the given points, say R, is a point at infinity. For then R determines a line direction l, and we take for M' the intersection with Γ of a line through M_2 parallel to l.

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; JORDI DOU, Barcelona, Spain; and J.T. GROENMAN, Arnhem, The Netherlands. Comments were received from LEON BANKOFF, Los Angeles, California; O. BOTTEMA, Delft, The Netherlands; RAM REKHA TIWARI, Radhaur, Bihar, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

Editor's comment.

The following dual problem can be solved in a similar way: *Circumscribe to a given conic a triangle whose vertices lie on three given lines.*

The effortless elegance of our featured solution shows that this construction problem is essentially of a projective nature, and that any attempt to force it within the straightjacket of Euclidean geometry is likely to be strenuously resisted. This is why our featured solver, in a covering letter, expressed surprise that the problem is to be found in Hall & Stevens's very elementary textbook on Euclidean geometry.

Yet the problem antedates projective geometry, and many famous geometers of the past have struggled with it. The information in the next paragraph was culled from Dörrie [2] and F. G.-M. [3].

The problem had been solved by Pappus for a given circle and three collinear points. Gabriel Cramer (1704-1752) proposed it for a circle and three arbitrary (ordinary) points. This version was first solved by Castillon in 1776, twenty-four years after Cramer's death (J.-F. Salvemini (1709-1791) took the name Castillon after his birthplace Castiglione in Tuscany), and the problem has come to be known as Castillon's Problem. (And Hall & Stevens ask rank beginners to solve it for next Thursday!) Among other mathematicians who subsequently published solutions are Lagrange, Malfatti, Lhuilier, Servois, Poncelet, and the Neapolitan Annibale Giordano Di Ottaiano. Both Dörrie and F. G.-M. give Giordano's solution, considering it to be the best. F. G.-M. says it is "la plus élégante", but he has to break it up into three separate problems (over two full pages of small print) to make it understandable. Dörrie starts by saying that it is a "simple, though not easily seen, solution"; then, after struggling with it for nearly three pages, he ends by calling it a "fairly intricate solution", and makes amends by giving a simpler projective solution by Jakob Steiner (1796-1863), roughly equivalent to our featured solution (but still, from the context, only for a circle and three ordinary points).

The problem as given here appears also in Hall & Stevens's *A Text-Book of Euclid's Elements* [4], and their solution appears in the *Key* to the exercises in their two textbooks (see [5] and [6]). For completeness (and for comic relief), we now give their construction and proof as it appears in [6], a copy of which was sent by Leon Bankoff. We have not deviated by one iota from the authors' quaint use of mathematical "English". (The internal references are to [4].) Ready?

Let XY be the given st. line, and P, Q the given pts. Join PQ and in it take a pt. F so that rect. PF, PQ = the rectangle contained by the segments of any chord of the circle through P [VI. 12]. Let QP and YX be produced to meet at Z. Let K be the length of a chord of the ⊙ which subtends at the o^{ce} an angle equal to $\angle$QZY; through F draw a line FBD cutting off a chord BD equal to K [Ex. 9, p. 197]. Draw PBA meeting ⊙ in B, A, and join AQ meeting the ⊙ in C. Then ABC shall be the required Δ.

Because rect. PF, PQ = rect. PB, PA:

$\therefore$ PF : PB = PA : PQ,

$\therefore$ Δ^s PBF, PAQ are similar [VI. 6].

$\therefore$ $\angle$PFB = $\angle$PAC

= $\angle$BDC, (or the supplement of BDC;)

$\therefore$ DC is par^l. to PQ.

And because $\angle$DCB = $\angle$QZY;

$\therefore$BC is par^l. to XY.

Here, mercifully, endeth the quotation. All but the most unreconstructed and blinkered disciples of Euclid will agree that, compared with our projective solution, this Euclidean solution, which gives no indication that there are in general two solution triangles, is terrible. And not just plain terrible: it is fancy terrible, it is terrible with raisins in it.

Render therefore unto Poncelet the things which are Poncelet's; and unto Euclid the things that are Euclid's. (After *Matthew* 22:21.) That we should here render unto Poncelet is made even more obvious by the fact that, while all Euclidean solutions make fanciful use of ruler *and* compass, a ruler suffices for our simple projective solution of the Castillon Problem. As Shakespeare wrote prophetically in *Antony and Cleopatra* (II, iii, 6),

> I have not kept my square; but that to come
> Shall all be done by the rule.

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AGE CANNOT WITHER...DONALD COXETER

Ever have trouble remembering your age?

Your problems are over, says author and teacher *Donald Coxeter*, a mathematics whiz at U of T. Coxeter, a classic absent-minded professor, endorses this simple formula:

"Take your house number and double it. Add 5. Multiply by 50. Then, add your age, the number of days in a year, and subtract 615.

The last two figures will be your age; the others your house number."

(From the *Toronto Star*, February 3, 1983.)

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