Hardy-Ramanujan Journal 42 (2019), 4-11

Remembrance of Alan Baker

Gisbert Wüstholz

It was at the conference on *Diophantische Approximationen* organized by Theodor Schneider which took place at Oberwolfach Juli 14 - Juli 20 1974 when I first met Alan Baker. At the time I was together with Hans Peter Schlickewei a graduate student of Schneider at the Albert Ludwigs University at Freiburg i. Br.. We both were invited to the conference to help Schneider organize the conference in practical matters. In particular we were determined to make the speakers write down an abstract of their talks in the Vortagsbuch which was considered to be an important historical document for the future. Indeed if you open the webpage of the Mathematisches Forschungsinstitut Oberwolfach you find a link to the Oberwolfach Digital Archive where you can find all these Documents from 1944-2008.

At this conference I was very much impressed and touched by seeing the Fields Medallist Baker whose work I had studied in the past year to do the p-adic analogue of Baker's recent versions *A sharpening of the bounds for linear forms in logarithms I* together with the subsequent *Sharpening II*. This was done previously by Coates in 1969 but based on an older and much less precise version. I did not have much conversation with Baker then.

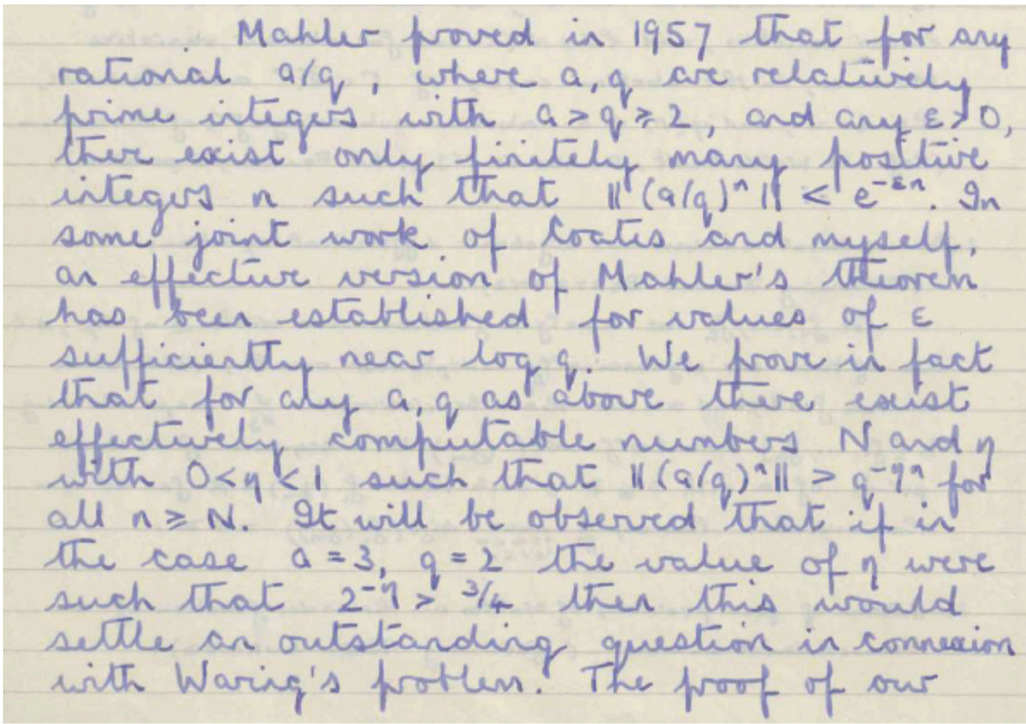
This was at the time when the old historical building, an old castle, of the Forschungsinstitut was still there.

Abbildung 9. Reichsinstitut für Mathematik (später Mathematisches Forschungsinstitut Oberwolfach)

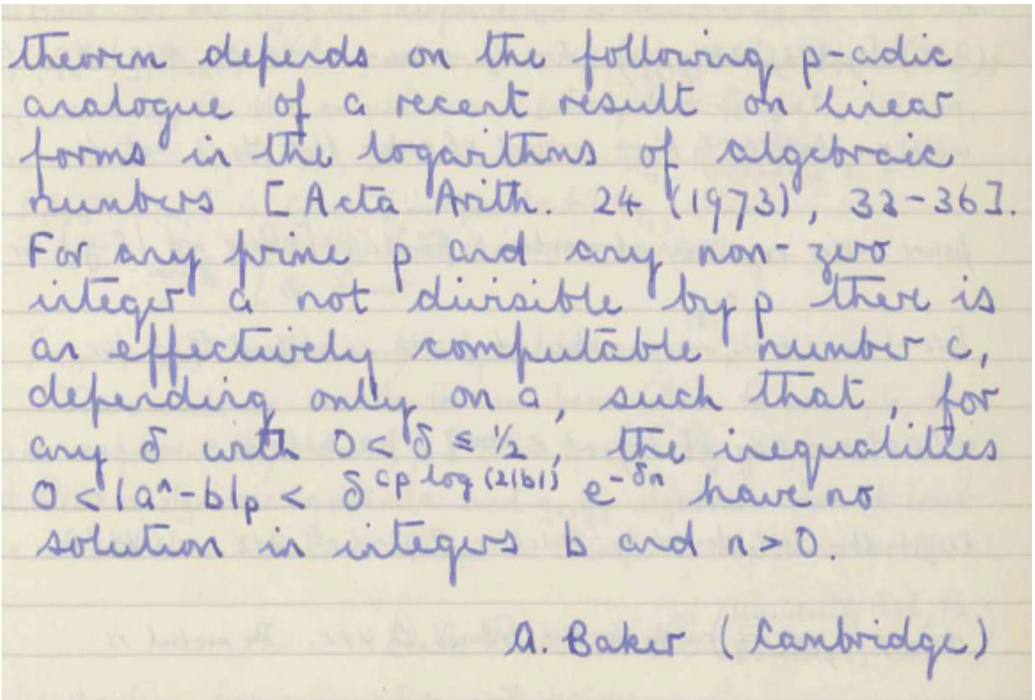
Later Baker told me many times that he misses the old building which was replaced in 1975 by the library and meetings building of the MFO, a gift of the VolkswagenStiftung. He liked the old castle and the atmosphere very much.

We thank episciences.org for providing open access hosting of the electronic journal *Hardy-Ramanujan Journal*

Baker gave a talk at this conference about joint work with Coates on an effective version of a result which has been obtained by Mahler in 1957. He also makes the remark that an improvement of the result of a special kind would settle an outstanding question in connection with Waring's problem.



Mahler proved in 1957 that for any rational a/q, where a, q are relatively prime integers with $a > q \geq 2$, and any $\varepsilon > 0$, there exist only finitely many positive integers n such that $\|(a/q)^n\| < e^{-\varepsilon n}$. In some joint work of Coates and myself, an effective version of Mahler's theorem has been established for values of ε sufficiently near $\log q$. We prove in fact that for any a, q as above there exist effectively computable numbers N and η with $0<\eta<1$ such that $\|(a/q)^n\| > q^{-\eta n}$ for all $n \geq N$. It will be observed that if in the case $a=3$, $q=2$ the value of η were such that $2^{-\eta} > 3/4$ then this would settle an outstanding question in connexion with Waring's problem. The proof of our



theorem depends on the following p-adic analogue of a recent result on linear forms in the logarithms of algebraic numbers [Acta Arith. 24 (1973), 33-36]. For any prime p and any non-zero integer a not divisible by p there is an effectively computable number c, depending only on a, such that, for any δ with $0 < \delta \leq 1/2$, the inequalities $0 < |a^n - b|_p < \delta^{cp\log(2|b|)} e^{-\delta n}$ have no solution in integers b and $n > 0$.

A. Baker (Cambridge)

Baker's abstract 1974 Oberwolfach

The conference in 1974 was not the first Oberwolfach Conference Baker attended. As far as I could trace he was at the Forschungsinstitut the first time in 1965 where he gave a talk on three different topics. The first was on effective approximation of $\sqrt[3]{2}$ by rationals, one of the famous results in the framework of Roth's theorem. The second topic was on linear forms with integer coefficients in $\exp(r_0), \ldots, \exp(r_k)$ where $r_0, \ldots, r_k$ are distinct positive rationals. The last topic was on a linear form with real coefficients in three rational primes.

Some recent results in Diophantine Approximation

Recent results were discussed relating to three particular topics in Diophantine approximation.

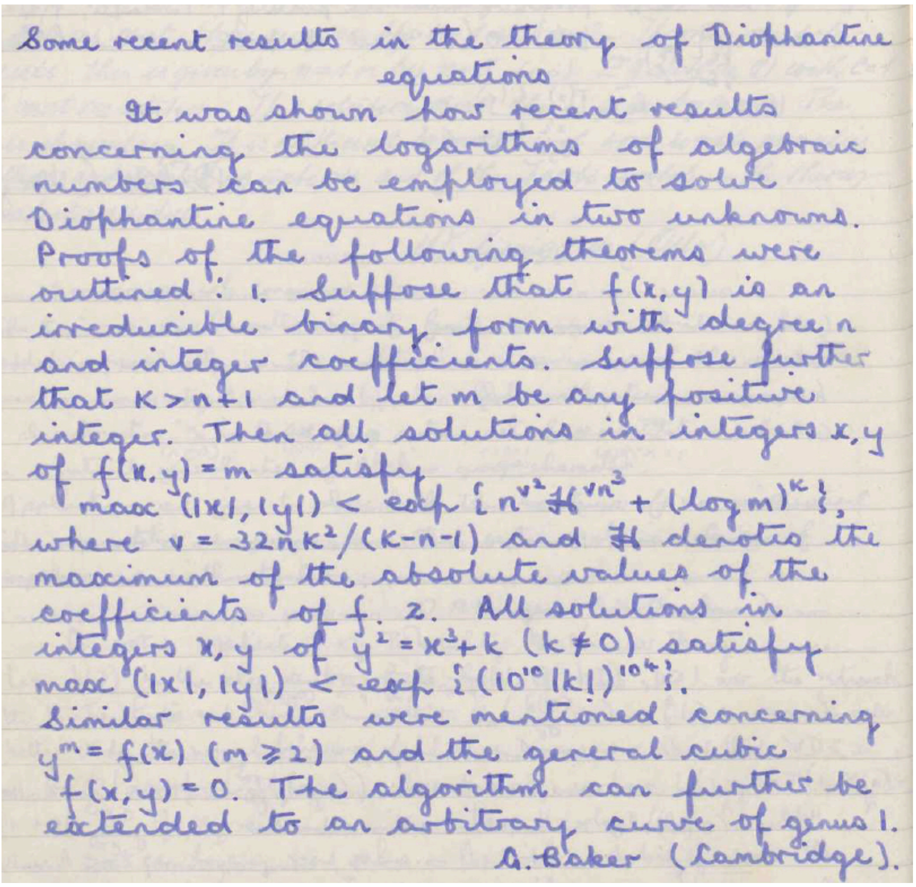
1. Suppose that α is an algebraic number, not rational, and that $\kappa > 2$. The Thue-Siegel-Roth Theorem implies the existence of $c = c(\alpha, \kappa) > 0$ such that $|\alpha - p/q| > cq^{-\kappa}$ for all rationals p/q $(q > 0)$ but, as is well known, the method of proof does not allow c to be determined explicitly. Effective results of this type were recently obtained for certain algebraic numbers α, given essentially by fractional powers of rationals, and, in particular, it was proved that for $\alpha = \sqrt[3]{2}$ and $\kappa = 2{\cdot}955$ we can take $c = 10^{-6}$. The work can be generalised to give new results concerning simultaneous rational approximations to certain sets of algebraic numbers.

2. After the original proof of the transcendence of e by Hermite in 1873 much work has been done to obtain successively better measures for this transcendence. Popken (1929) and Mahler (1932) proved, in particular, that there exist only finitely many sets of integers $x_0, x_1, \ldots, x_k$ such that $|x^{k+1}(x_0 + x_1 e + \ldots + x_k e^k)| < x^{1-\eta(x)}$, where $x = \max |x_i|$ and $\eta(x)$ is of order $(\log \log x)^{-1}$. It can now be shown that if $\theta_0, \theta_1, \ldots, \theta_k$ are positive numbers such that $\log \theta_0, \log \theta_1, \ldots, \log \theta_k$ are distinct rationals then there exist only finitely many sets of non-zero integers $x_0, x_1, \ldots, x_k$ such that $|x_0 x_1 \ldots x_k (x_0 \theta_0 + x_1 \theta_1 + \ldots + x_k \theta_k)| < x^{1-\varepsilon(x)}$, where $x = \max |x_i|$ and $\varepsilon(x)$ is of order $(\log \log x)^{-\frac{1}{2}}$. The special case $\theta_1 = 1, \theta_2 = e, \ldots, \theta_k = e^k$ provides a further measure of transcendence for e.

3. Let $\lambda_1, \lambda_2, \lambda_3$ be non-zero real numbers, not all of the same sign, with one at least of the ratios λ_i/λ_j irrational. Then it can be proved that for any positive integer n there exist infinitely many primes p_1, p_2, p_3 such that $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3| < (\log p)^{-n}$, where p denotes the maximum of p_1, p_2, p_3.

A. Baker

After this 1965-conference he started to develop his theory on linear forms in logarithms. As an outcome, he was able to solve a sequence of outstanding diophantine problems. The talk was one of the moments of glory in diophantine geometry and also at Oberwolfach. Another result which was not reported in Oberwolfach was the class number 1 problem. This



Some recent results in the theory of Diophantine equations

It was shown how recent results concerning the logarithms of algebraic numbers can be employed to solve Diophantine equations in two unknowns. Proofs of the following theorems were outlined: 1. Suppose that $f(x,y)$ is an irreducible binary form with degree n and integer coefficients. Suppose further that $\kappa > n+1$ and let m be any positive integer. Then all solutions in integers x, y of $f(x,y) = m$ satisfy

$$\max(|x|,|y|) < \exp\{n^{v^2} \mathcal{H}^{vn^3} + (\log m)^{\kappa}\},$$

where $v = 32n\kappa^2/(\kappa-n-1)$ and $\mathcal{H}$ denotes the maximum of the absolute values of the coefficients of f. 2. All solutions in integers x, y of $y^2 = x^3+k$ $(k \neq 0)$ satisfy $\max(|x|,|y|) < \exp\{(10^{10}|k|)^{10^4}\}$. Similar results were mentioned concerning $y^m = f(x)$ $(m \geq 2)$ and the general cubic $f(x,y) = 0$. The algorithm can further be extended to an arbitrary curve of genus 1.

A. Baker (Cambridge).

was attacked before by Heegner in a paper which was not accepted in the community. Baker solved it using linear forms in logarithms but the situation with Heegner's proof remained unclear at the time. Schlickewei wrote his Masters thesis with Schneider on Heegner's proof around that time.

All these results stimulated very strongly research on linear forms in logarithms. As I mentioned earlier I decided to work on linear forms in p-adic logarithms. As far as I can remember this was one of the topics for a PhD thesis which Schneider suggested to Schlickewei and myself when we had a first meeting on PhD problems. I started in 1973 and worked on it before the 1974 Oberwolfach conference. There I met van der Poorten whom I reported on the status of my work. To my surprise he himself gave a talk on lower bounds in linear forms in p-adic logarithms and claimed to have obtained the full p-adic analogue of Baker's theorem. I was very depressed because van der Poorten was considered as an authority and I gave up. Later it turned out that his proof contained serious mistakes which eventually were fixed in a first paper of my PhD student Kunrui Yu. I then changed my PhD problem and turned to linear forms in logarithms of U-numbers and got the PhD degree at Freiburg.

After the Oberwolfach conference in 1974 Baker stopped to attend the subsequent conferences and after 1977 he also stopped to work in a serious way on linear forms in logarithms. The main stream in transcendence moved from linear forms which seemed to have run into a dead end to group varieties and various problems in that direction were attacked. Eventually I succeeded to prove the analytic subgroup theorem which put the Baker theory onto a much higher and general level. New methods from algebraic geometry were needed to replace the more ad hoc tools in the Baker theory by multiplicity estimates on group varieties. This was partly developed jointly with Masser. As an easy corollary Baker's qualitative theorem on linear forms could be derived from the analytic subgroup theorem. Around 1985, to very much my surprise, I realized that the very abstract theory was able to remove a second order factor from the inequalities that Baker established in 1977 which was considered as the ultimate result provable by the methods at the time. The new ingredient was the use of our multiplicity estimates on group varieties applied to the group variety which is the product $\mathbb{G}_a \times \mathbb{G}_m^n$ of the additive group $\mathbb{G}_a$ and a product of multiplicative groups $\mathbb{G}_m$, now called the Baker group. Our result was effective but we did not give an explicit determination of the constant. I reported on this at the stimulating conference which Baker organized in 1987 in Durham. Baker was very much fascinated by the new result and asked the question to make it fully explicit. At the same time, Philippon and Waldschmidt also proved a similar result.

In this period Baker also invited me to give a talk in his number theory seminar which took place at that time in the old Department DPMMS, housed in a converted warehouse at 16 Mill Lane. After the talk there was a drink in the The Mill, a pub right next to the department. Later in the afternoon, there was traditionally a party in Baker's Apartment in Trinity College which was very famous. His bedmaker usually contributed with some homemade cake and everybody enjoyed the event. Also at this time I had the first opportunity to have dinner with Baker in the hall at Trinity College and after having entertainment in the Old Combination Room. To me this was like being for a moment in the middle ages. Later I was very often dining with Baker there and also staying in the Old Combination Room with old Port and Stilton cheese. Of course it was necessary to be dressed in a formal way with the fellows waring the gown. On a regular schedule on Sunday evening the master of Trinity opened the Loge for enjoying social life after dinner.

Also lunch was very special but not so formal. Baker took me there quite regularly whenever I visited him in all the later years. Having finished with the lunch there was coffee in the senior common room. Baker occasionally met some of his friends there and when the weather was fine, i.e., no rain and a glance of sun, we went for bowling in the Fellows' Bowling Green. Bowling at Trinity is very special. Bowls are not bowls but some sort of wooden discs with unequal boundary so that the disc would not roll as one would expect. The fellows had great fun and as far as I remember I also won one time.

In 1987, I moved from the Bergische Gesamthochschule Wuppertal to ETH Zurich. As one of my first activities, I invited Baker to stay for a couple of months at ETH and to give a graduate course at our Department in the winter term 1988-89. On this occasion, I took up Baker's question he raised at the Durham conference in 1987, namely to make the constant in the lower bound I had derived earlier fully explicit. This would be clearly a very tedious work, in particular if one intends to get an optimal result. I suggested Baker to do this jointly and he was very anxious to begin. It took us then more than 3 years to finish. And we not only went through my proof and kept track of the constants. On the contrary, we developed new strategies to optimize the proof with respect to the constants. The new key ingredients were the combination of very fine multiplicity estimates especially adapted to the Baker group, a special use of geometry of numbers and a new application of Kummer theory which was Baker's ultimate tool in his last 1977 paper on logarithmic forms but which failed to eliminate the second order term. This made it necessary to run through a very complicated and tedious double induction process. To work this out took us much energy and also some intense fights but without shuttering our friendship. Within the three years of work we not only met in Zurich but also in Cambridge. Baker invited me in 1992 as a visiting fellow commoner at Trinity for a couple of months with the aim to get the joint work done. He organized a very nice arrangement for housing in the College. Just opposite to the Great Gate in Whewell's Court he got me the apartment overlooking the gate. He told me that Hardy had stayed there for quite a while. As a Visiting fellow commoner I had almost all rights of a fellow including crossing the greens, having dining rights and the right of buying vintage port and claret from the College's cellar. In the same year I was proposed by Baker to give the Mordell lecture, a very prestigious lecture with a reception after the talk in the bowling greens.

After having finished the paper Baker suggested to write a new updated version of his book on transcendental numbers which at the time became outdated and not representing the new exciting developments anymore. It turned out to become again a long term project. Baker was very unhappy that it took so long. In particular because Atiyah was regularly kidding him because of it seemed to him that it would never appear. After the paper on logarithmic forms was finished Baker came several times for extended visits to the Forschungsinstitut FIM attached to the Department of Mathematics at ETH and I visited Trinity College also a few times. The book finally appeared in 2008 but it was still a long way to go there.

In the preface of the book we write *This book has arisen from lectures given by the first author at ETH Zürich in the Wintersemester 1988-89 under the Nachdiplomvorlesung program and subsequent lectures by both authors in various localities, in particular at an instructional conference organized by the DMV in Blaubeuren. Our object has been to give an account of the theory of linear forms in the logarithms of algebraic numbers with special emphasis on the important developments of the past twenty-five years concerning multiplicity estimates on group varieties. As will be clear from the text there is now much interplay between studies on logarithmic forms and deep aspects of arithmetic algebraic geometry. New light has been shed for instance on the famous conjectures of Tate and Shafarevich relating to abelian varieties and the associated celebrated discoveries of Faltings, establishing the Mordell conjecture. We give a connected exposition reflecting these major advances including the first version in book form of the basic works of Masser and Wüstholz on zero estimates on group varieties, the analytic subgroup theorem and their applications. Our discussion here is more algebraic in character than the original and involves, in particular, Hilbert functions in degree theory and Poincaré series as well as the general background of Lie algebras and group varieties.*

One of the basic instances to begin with for writing the book was our visit at the University of Hong Kong in 1999. We talked a lot about logarithmic form there and, on the suggestion of Kai Man Tsang, organized a conference. We also had several discussions about the *abc* conjecture. We both enjoyed staying there for a longer period. Baker liked to spend every night somewhere in down town Hong Kong to have dinner and drinks there.

His father was a tailor and naturally Alan liked tailored dressing. Hong Kong is a perfect place for that and during our joint visit there he let tailor stitch a new suit. He swarmed from the many fittings so that I myself also went to a place to get a jacket tailored. This caused much gossip between us and made the life there very pleasant and entertaining. We also talked about the conference on the occasion of Baker's 60th birthday which I was planning together with the FIM to be held the same year. It took place during 30 August to 4 September, 1999 just after his birthday on August 19.

A very illustrious group of world class speakers came together to celebrate the birthday including three other Fields Medallists, Bombieri, Faltings and Margulis. Baker felt very honoured and at the conference dinner which took place at the FIFA restaurant on the Sonnenberg at Zürich, a very elegant place overlooking Zürich, he even gave an extended speech which was a panoramic look over his life. I don't remember the details but David Masser later reminded me that at one point in his speech he admitted that he felt very much sad not having the opportunity to marry. I edited a proceeding of the conference which appeared in 2002 in Cambridge University Press. Its title – A Panorama in Number Theory *or* The View from Baker's Garden – indicates the intention behind which can be read off from the preface from which I quote briefly *The millennium, together with Alan Baker's 60th birthday offered a singular occasion to organize a meeting in number theory and to bring together a leading group of international researchers in the field; it was generously supported by ETH Zürich together with the Forschungsinstitut für Mathematik. This encouraged us to work out a program that aimed to cover a large spectrum of number theory and related geometry with particular emphasis on Diophantine aspects.*

In the time after this conference he came very regularly to ETH to work with me on the book. I also visited Trinity maybe two times for a longer visit. At one occasion I gave one of the Kuwait Foundation Lecture on February 12, 2001. My part in the book project was to write the chapters 4 to 8 and since I did not have any written notes, much of the work was original research work and, consequently, proceeded rather slowly. We had discussions about each sentence, and even about each word.

During these visits the secretaries always found a nice accommodation for him. Once the apartment he was offered was very special. It was in an old historical building in the old part of Zurich not far away from ETH and located in the Spiegelgasse 14 where Lenin stayed from February 21, 1916 till April 2, 1917. The apartment was very nice and in the basement located with a tiny hidden garden in the back. Baker to my very much surprise turned it down because it had only a shower but no bath tube. Not far away in Spiegelgasse 1 is the famous Cabaret Voltaire, an artistic nightclub where the anarchic art movement known as Dada was founded. At another occasion he was lodged in an apartment in Zurich Höngg. One day the landlord gave a phone call to the secretary complaining that the strange professor was shouting for a good while in the middle of the night in his room so that he could hear him and was disturbed. It was then the embarrassing task of the secretary to communicate this to Baker. He was famous for such self speeches at times normal people sleep. Also in his apartment at Trinity one could hear him from time to time. Generally speaking he was very charming but also not always easy. For instance in tax questions which had to be settled by FIM he was very fussy. Not only did he minimize constants in lower bounds for linear forms in logarithms but also in tax duties.

The book was finally completed in 2006 and appeared in 2008 in the Cambridge University Press series New Mathematical Monographs. Baker was very happy about it and we celebrated the event in one of the special rooms in the college with a group of friends and colleagues. In some sense I think he regarded it as some sort of his mathematical legacy. At the time Baker had to retire. He was not happy about this at all. This added to several other instances regarding the department in the past. Already when he got the Fields Medal in 1970 he had a very minor position at the department and was then only promoted in a minor way before he finally got a personal chair in 1974. The Royal Society was much faster. He was elected as FRS in 1973 not even being a professor. Later around 1986 he hoped to get a named chair but things developed differently. He spoke about this very often when we were out for dinner and talked about this and that and it was then difficult to change the conversation. From this time on he almost stopped research until he started joint work with me around 1990.

During the time we stayed at Hong Kong University in 1999 we also were talking about the *abc* conjecture and after the birthday conference we began to look at it more carefully. We did not attempt

Alan Baker and Kunrui Yu enjoying the appetizer at Belvoir Park to prove it but we wanted to see how close the conjecture becomes to the reality. Baker finished three publications between 2000 and 2010 with very interesting insights. Then in 2009 FIM organized a conference celebrating my 61st birthday and Baker was one of the main speakers together with Kunrui Yu. The conference dinner was at a very nice restaurant in Zurich located in the beautiful Belvoir Park where we had an appetizer. Later in the evening he enjoyed the wine and the conversation with Faltings.

Alan Baker and Gert Faltings enjoying the drinks at Wüstholz 61

After the conference Baker somehow retired from active mathematics and I hardly could get him to accept the invitation to the conference 25 Years of Number Theory Seminar at ETHZ. One of the problems Baker had was that his passport had expired probably since several years before that conference. He did not travel all these years although traveling was besides playing table tennis, theatre and bowling one of his favourite hobbies. Besides Faltings and Bombieri he was one of the most frequent visitors at FIM during these 25 years. He gave a talk but one could feel that he was going to get tired, maybe he got depressed and since then our contact stopped until shortly before his passing away when we had a last very friendly exchange of emails.

Gisbert Wüstholz
Departement of Mathematik
ETH Zurich, Rämistrasse 101
CH-8092 Zürich. Switzerland
e-mail: gisbert.wuestholz@math.ethz.ch