



“Before Everything was Completely Clear, We Had a Few Arguments.” - Emmy Noether, David Hilbert and Felix Klein

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Abstract

This article reconstructs the historic events and records leading up to Emmy Noether’s famous paper on “Invariant Variational Problems” (1918). Based on the available archival evidence, especially the extensive manuscript notes by Felix Klein, it examines the intensive collaboration between Emmy Noether, David Hilbert and Felix Klein in Göttingen in the period between 1915 and 1918. This reflection shows how Noether’s paper emerged from a discussion context in a historical setting, shaped by both supportive and obstructive circumstances that influenced the development of her approach.

1 Introduction

Emmy Noether (1882–1935) is renowned for her pioneering work in the foundations of modern algebra as well as for a number of outstanding achievements in pure mathematics.¹ In physics, her name is lauded for two theorems, proven and published in a paper on “Invariant Variational Problems” [30]. These theorems gained wide-ranging significance as they established a deep and general connection between symmetries and conservation laws in science.² This groundbreaking work also served as Noether’s habilitation thesis, a formal post-doctoral process in German academia to obtain the *venia legendi*, the authorization to teach independently at a university. While Emmy Noether became both the first woman to be granted the title *Privatdozentin* (private lecturer) at the University of Göttingen in 1919 as well as the first

¹ For recent biographies, see [20, 40], and [39].

² [23] and [33], see also [2, 35–38, 56].

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woman ever to obtain the *venia legendi* in mathematics at a German-language university, she only succeeded at the third attempt. Two previous habilitation applications had been rejected, as a a decree from 1908 expressly prohibited women in Prussia from habilitation.³ Neither her academic promotion in Göttingen to private lecturer nor to associate professor in 1922 came with a salary. Despite her successful teaching and the international acclaim she had garnered,⁴ she was expelled from the university by the Nazis in 1933. Of Jewish descent, she, like many other scientists at the time, emigrated to the United States, where she died in 1935 from complications of a surgical procedure.

Mathematical work, like all science, occurs in a social environment, where mathematicians interact, debate, challenge and influence each other. Historical analysis must consider this context. Even a cursory glance at Emmy Noether's seminal work on "Invariant Variational Problems" reveals that it emerged from a discussion context. Presented to the *Göttingen Academy of Sciences* for publication in its Proceedings by Felix Klein (1849–1925) on July 26, 1918, the submission date footnote indicates: "The final version of the manuscript was not submitted until the end of September." In the first introduction section, Noether references several relevant works, including Felix Klein's, stating: "In particular, Klein's second paper (termed *Second Note*) and the present publication submission are influenced by each other, for which I may refer to the concluding remarks of Klein's paper." Klein's *Second Note* [17] had been presented to the academy just a week earlier, on 19 July, likewise with a note adding that the the final version was only received "mid-September." In his concluding remarks, Klein acknowledged Noether: "I must also not omit to thank Miss Noether again for her supportive participation in my new works, who in turn worked out in generality the mathematical ideas, that I have employed in adaptation to the physical problem [...]." While these two works by Noether and Klein explicitly reference each other, Noether also discusses what she termed a "Hilbertian assertion" [30, § 6], which she thoroughly explains and proves, in the last section of her article.

Overall, there exists a sizable set of manuscripts by Felix Klein from the relevant period that contains numerous references to the discussions between himself and Emmy Noether. Said manuscripts, together with select references in David Hilbert's (1862–1943) literary estate, allow for a detailed picture of the interactions between Emmy Noether and her Göttingen colleagues. The aim of this article is to shed light on the interactions between Emmy Noether and both Felix Klein and David Hilbert, drawing upon the available archival documents. These sources have also been thoroughly analyzed in great detail and with many extensive quotations and facsimiles from the original sources in a recent volume by Cordula Tollmien [59]. This work represents the third volume of Tollmien's ongoing biographical study on Emmy Noether's life and contributions, serving as a valuable resource for those scholarly interested in the subject.⁵

³ See [54, 57, 58], see also [48, 49, 44, 15].

⁴ [22].

⁵ See also [57, 58].

2 Prehistory

Emmy Noether first came to Göttingen as a student for a semester in 1903. She returned for a longer residence (punctuated by returns to her hometown of Erlangen) from April 1915 until 1933, when she emigrated to the United States, and was active in mathematical discussions there [40]. Noether's connections to Felix Klein predated her first arrival in Göttingen, stemming from the long-standing friendship between Klein and her father Max Noether (1844–1921) [51, 41–42]. The three co-authored an obituary for Paul Gordan (1837–1912) in 1912 (see [53]; [32]).

Noether's personal contact with David Hilbert began during her later residence in Göttingen, growing out of their shared interest in invariant theory. Only two letters from Noether to Hilbert are preserved in the Hilbert estate.⁶ Both are submissions for publication in the *Mathematische Annalen*, which Hilbert edited. The first of these, dated May 4, 1914, explains that Noether's submission on "Fields and Systems of Rational Functions" [28] builds on Hilbert's [12] "On the Full Invariant Systems" as well as on "the problem of "Relative-Whole Functions,' problem 14 of your mathematical problems."⁷

Her second letter to Hilbert, dated December 1, 1914, also refers to a work by Hilbert [13]:

In the paper 'On the Invariants of a System of any Number of Fundamental Forms' sent to my father yesterday, you express the conjecture *that these invariants can be represented completely and rationally by the finitely many invariants of the system (I, PI)* . You may therefore be interested to know that [...] this conjecture can indeed be easily proven.⁸

On top of providing a sketch of this proof, Noether uses the opportunity to announce her submission plans of "a short manuscript" [29]. She explains that it discusses the work of Ernst Zermelo (1871–1953) and anticipates submission within the next few weeks. Again, she outlines her results and informs: "I already communicated these results in detail to Mr. Zermelo at the end of October; however, I did not receive a reply. My two letters may not have passed the border control. (ibid.)"

When Klein and Hilbert became aware of Noether's unusual mathematical talent, they invited her to Göttingen where the group of mathematicians had been severely decimated by the war. As Cordula Tollmien [54, 57, 58] has meticulously reconstructed, Noether pursued obtaining her habilitation in Göttingen from the very beginning. Her unusual action defied the existing ban on academic teaching by female lecturers. During the summer semester of 1915, Noether gave a lecture on "Questions of Finitude in Invariant Theory" on July 13 at the *Göttingen Mathematical Society*. A week later, on July 20, she officially applied for her habilitation. The habilitation committee met the following day, and Emmy Noether returned to Erlangen on July 31st.

⁶Staats- und Universitätsbibliothek Niedersachsen (SUB) Cod.Ms.Hilbert 284.

⁷SUB Cod.Ms.Hilbert 284/1; published in [21, 58, pp. 33–40].

⁸SUB Cod.Ms.Hilbert 284/1; published in [20, pp. 17–18], [58, pp. 28–29].

Noether's habilitation application received general support from the Göttingen mathematicians and physicists. As Tollmien observed, though, Hilbert was unique in focussing solely on Noether's mathematical achievements in his supporting statement. From the beginning, Noether's entry into the Göttingen mathematician community was marked by a precarious academic affiliation. This circumstance remains crucial to our understanding, even if the following presents itself primarily as a reconstruction of a professional debate.

Upon her return to Göttingen for the winter semester of 1915/16, Noether encountered an atmosphere of intense scientific activity. Hilbert and Albert Einstein (1879–1955) competed against each other in the final steps of establishing the general theory of relativity. This sequence of events has been extensively discussed in the literature,⁹ thus, only essential steps will be recounted herein.

In the summer of 1913, Einstein and Marcel Grossmann (1878–1936), his student friend and then mathematics peer at the ETH, had published an “Outline of a Generalized Theory of Relativity and of a Theory of Gravitation” [7]. It presented a relativistic theory of gravitation which was already formulated in generally covariant form in almost all its parts. But it still advanced field equations, whose covariance group was unknown but restricted. A second joint publication of May 1914 entitled “Covariance Properties of the Field Equations of the Theory of Gravitation Based on the Generalized Theory of Relativity” [8] gave a variational formulation of these equations. It was Paul Bernays (1888–1977) in Göttingen, who had suggested the variational approach (*ibid.*, p. 219). It allowed them to investigate the open question of the covariance group of their “outline” equations.

Einstein's and Grossmann's work picqued the interest of mathematicians in Göttingen, especially Hilbert, who invited Einstein to give a series of six lectures on the latest state of his theory of gravitation. These lectures took place in the week from June 28 to July 5, a week before Noether's talk at the *Göttingen Mathematical Society*. Clearly, she was one of the participants in Einstein's lectures [3, pp. 320–323] and was undoubtedly meant when Einstein wrote to Arnold Sommerfeld (1868–1951): “In Göttingen I had the great pleasure of seeing everything understood down to the last detail.”¹⁰

The final field equations of the general theory of relativity were discovered in November 1915. Here we highlight only the most important dates. Einstein presented four communications to the *Prussian Academy of Sciences* in Berlin at weekly intervals for publication in its *Proceedings*, namely on November 4, 11, 18 and 25. The last of these contained Einstein's final field equations. Hilbert, on the other hand, presented his own paper, entitled “Foundations of Physics. First Communication.” [14], to the *Göttingen Academy of Sciences* on November 20. However, the text of his paper was substantially changed after December 6 [4].

Notably, Noether's habilitation application was widely discussed during the same time span: on November 6 in the mathematical-physical department of the Faculty of Philosophy, on November 10 in the historical-philosophical department, and on November 18 in a meeting of the entire faculty. Subsequently, on November 26, the

⁹See, e.g., [41, 34] and further literature cited therein.

¹⁰Einstein to Sommerfeld, 15.7.1915, [45, Doc. 96].

faculty sent a request for dispensation from the habilitation ban in the individual case of Noether. On December 4, right before the university chancellor forwarded this request to the Prussian Minister of Education on December 8, Hilbert petitioned directly to the minister:

“[...] the mathematical-physical progress (Einstein’s theory of gravitation, theory of time and space) is also striving towards an undreamt of culmination; and here I have Miss Emmy Noether as my most successful collaborator. (quoted after [54, p. 22])

Klein’s notes, to be discussed in more detail below, reveal that Emmy Noether and Felix Klein met and deliberated general questions on November 28, 1915 and discussed general questions, including those concerning Hilbert’s problems (SUB Cod. Ms.Klein 22B, pp. 78ff). Could this exchange possibly be connected to preparations for Noether’s habilitation lecture?¹¹ More likely, Klein and Noether conferred more generally about the recent development in mathematics in general. This would have corresponded to Klein’s interests in planning his lecture series on the historical development of mathematics.

In any case, Klein’s records disclose his annotation “With Miss N.” adjacent to the date under the heading “Since 1900”. It appears that Klein and Noether touched upon a wide array of the latest questions in contemporary mathematics. First, the subject was “philosophy”. Henri Poincaré (1854–1912) is referenced, likely in connection with non-Euclidean geometry, alongside Einstein’s new concept of space and time; the terms “Axiomatics and Pure Nominalism” are noted, as well as “Zermelo’s Treatment of Set Theory” and the inquiry, “To what extent has the concept of the integer been deepened?” They also investigated questions of “Arithmetic and Algebra.” The topics encompassed the “Status of the Class Field”, works by Klein’s and Hilbert’s students, i.e. Philipp Furtwängler (1869–1940), Felix Bernstein (1878–1956), Erich Hecke (1887–1947), and Rudolf Fueter (1880–1950). Hilbert’s own work on Waring’s theorem and an “extension to ∞ many variables (for transcendental questions)” was deliberated, too. Furthermore, the “Calculus of Real Variables” was thoroughly revealed, relating to Henri Lebesgue’s (1875–1941) work from 1904, as well as integral equations, the calculus of variations, and Dirichlet’s principle. Finally, they also explored questions of physics: “Math. physics. Order of phys. Ideas through mathematics.” and “Riemann’s ds^2 .” Regrettably, this discussion between Klein and Noether is only documented fragmentarily by Klein’s keywords.

¹¹ Cordula Tollmien pointed out to me that, contrary to this assumption, Noether had already attached three topics for a trial lecture to her application for habilitation in July 1915, although, unfortunately, a list of appendices to the application from November 1915 does not mention the topics [58, pp. 22–24]: cf. also her discussion of these notes in [59, pp. 252–257].

3 The Collaboration Between Emmy Noether and David Hilbert

During the early phase of her career in Göttingen, Emmy Noether appears to have primarily interacted with David Hilbert. Unfortunately, there are limited direct sources available to reconstruct the nature of their collaboration. One such document was found in Hilbert's estate in Göttingen. To fully appreciate this document, a more comprehensive technical background is necessary.

Hilbert had identified the interplay between invariant theory and calculus of variations as key in the early version of his 1915 paper (also termed *First Communication*). Invariant theory was one of Emmy Noether's areas of expertise, whereas calculus of variations was predominantly pursued by Constantin Carathéodory (1873–1950) in Göttingen.¹² To Hilbert, the solution to the problem of a general theory of relativity, which would be both a theory of gravitation and a theory of electromagnetism, was directly derived from the mathematical framework of a variational principle.

It was well established that Maxwell's field equations could be derived from a variational formulation. Hilbert referenced a recent generalization of Maxwellian electrodynamics proposed by Gustav Mie (1868-1957). Mie [25–27] suggested that the underlying variational principle should be generalized by permitting arbitrary terms in the action integral, provided it remained an invariant under Lorentz transformations, i.e. under special-orthogonal linear transformations of the space-time variables. Hilbert's synthesis of Mie's electrodynamics with Einstein's general theory of relativity was based on a variational principle that should remain invariant not only under linear transformations, but under general coordinate transformations. Nevertheless, the context of Mie's electrodynamics still imposed certain restrictions on variational integrals to ensure they were physically permissible.

In the context of a generally covariant variational formulation a well-defined problem arose: which action integrals could serve as the foundation for a physical theory? Hilbert appears to have posed a purely mathematical inquiry to Emmy Noether regarding the types of invariants that could be incorporated into the variational integral if Mie's restriction were lifted. This question is directly related to invariant theory, the field in which Noether was a recognized expert. The solution is documented in a note found within Hilbert's estate, written in Noether's distinctive German cursive script (SUB Cod.Ms.Hilbert 634, Bl.6). Although the note is undated, contextual evidence suggests it was composed shortly before Hilbert's writing of his *First Communication*. Noether's response seems to have encouraged Hilbert to focus on the most general case for physical theorizing, even within Mie's constraints.

In his *First Communication* of November 1915, Hilbert formulated a special case of the later (second) Noether theorem. It can already be found in the earlier, later revised typesetter version of his paper, as well as in the published version, although in neither version with proof. It says:

¹²Carathéodory had written both his doctoral and habilitation theses in Göttingen on problems of the calculus of variations and would later also write a textbook on that subject. Hilbert and Carathéodory spoke on November 30 and December 7 at the Göttingen Mathematical Society on "Invariant Theory".

The leitmotif for the structure of my theory is provided by the following mathematical theorem, the proof of which I will present elsewhere.

Theorem I. If J is an invariant under any transformation of the four world parameters, containing n quantities and their derivatives, and if one then forms the n Lagrangian variational equations with respect to those n quantities from

$$\delta \int J \sqrt{g} d\omega = 0, \quad (1)$$

then four equations in this system of n differential equations for the n quantities are always a consequence of the $n - 4$ others — in the sense that four linear, mutually independent combinations of the n differential equations and their total derivatives are always identically fulfilled. [14, p. 397]

Hilbert referred to this theorem as the “leitmotif” of his theoretical framework. He considered the ten gravitational equations, resulting from variation with respect to the metric, and the four electromagnetic field equations, derived by variation with respect to the electromagnetic potential, to be interdependent. Specifically, he posited that the electromagnetic equations were, in a particular sense, a direct consequence of the gravitational equations. He stated: As a result of the theorem established above, the four equations (5) [i.e. the electromagnetic equations] can be regarded as a consequence of equations (4) [i.e. the gravitational equations], i.e. we can state directly on the basis of that mathematical theorem that, in the sense indicated, the electrodynamic phenomena are effects of gravitation. [14, 397]

Hilbert fundamentally revised his theory of November 1915, partly in response to the inherent challenges of his approach, and partly in reaction to Einstein’s concurrent breakthrough on generally covariant field equations. The distinction between an early version, survived as galley proofs, and the final published version reflects this revision. This consequential revision has been discussed at length in the historical literature.¹³

During these deliberations, Hilbert delivered a presentation on “Invariant Theory and the General Energy Theorem” to the *Göttingen Mathematical Society* on January 25, 1916. A week later, on February 1, 1916, Emmy Noether presented on “Alternatives in Non-linear Systems of Equations” [23, p. 167].

All of this suggests that Emmy Noether was indeed an equal discussion partner in Göttingen during this pivotal phase in the development of the general theory of relativity. In their discourse, Hilbert and Klein appeared to have engaged Noether as an expert on pure mathematics, in particular, on invariant theory. This is noteworthy since Hilbert and Klein, for their part, advocated a mathematical approach in the

¹³ See, e.g., [4, 41] and further literature cited therein.

development of the theory of relativity, in contrast to Einstein’s more physically oriented reasoning.

Hilbert, who had been lecturing on mathematical physics for several semesters, and often incorporated his own current research interests into his lectures, addressed relativity theory in the summer semester of 1916. He announced this course, as well as his article, under the titles “Foundations of Physics.” At the commencement of this lecture course, Hilbert explained:

The lecture that I have announced this semester as “Foundations of Physics” is essentially intended to elaborate on modern relativistic ideas, which can probably claim the name “Foundations” with just as much, if not more, justification as other modern theories and ideas. [43, S.81]

Hilbert aimed for special relativity first, then for general relativity, adding:

Our mathematical tools here will be the calculus of variations and invariant theory. The latter will be covered by Dr. Noether in a series of lectures in the seminar. (ibid.)

Announcing Noether’s teaching in this peculiar way reflects the fact that Hilbert and Klein had not succeeded in pushing Emmy Noether’s habilitation through the institutional barriers against women’s habilitation. She was therefore only able to lecture under Hilbert’s name. In fact, we find the following entry in the University of Göttingen’s lecture catalog for the winter semester 1916/17:

Invariant Theory: Prof. Hilbert with the assistance of Miss Dr. E. Nöther, Mondays 4–6 p.m., free of charge.

Such postings of Emmy Noether’s teaching activities are regularly found in the course catalogs of subsequent semesters.

For our purpose, it is interesting that Hilbert introduces Noether as an expert for the relevant mathematical tools by explicitly communicating that she would cover invariant theory, one of the two relevant mathematical sub-disciplines for the general theory of relativity, independently on her own.

Further insights into the collaboration between Noether and Hilbert can be extrapolated from a brief exchange of letters between Albert Einstein and Hilbert from the end of May 1916. On May 25, Einstein wrote to Hilbert, “I am sitting over your relativity paper, which I must review in Rubens’ colloquium, and am honestly toiling over it.” Einstein had closely studied Hilbert’s arguments but remained unclear on certain points, including the energy theorem, noting:

I do not yet understand the energy theorem, not even as an assertion.[45, Doc. 221]

Seeking clarification, he requested “kind enlightenment”.

Hilbert responded three days later in a letter dated May 27, 1916. Addressing Einstein's first questions, he declared:

My energy theorem is probably related to yours: I have already assigned this question to Miss Noether. [45, Doc. 222]

Hilbert further addressed one of Einstein's objections and added: For brevity's sake I send you the enclosed snippet by Miss Nöther. (ibid.)

Unfortunately, the enclosed note has not been preserved, but clues can be found in Einstein's reply dated May 30, in which he expressed gratitude for Hilbert's explanations. In regards to the energy theorem, however, he commented: In your paper everything is understandable to me now except for the energy theorem. Please do not be angry with me that I ask you about this again. [45, Doc.223]

Specifying his question, he concluded: How is this cleared up? It would suffice, of course, if you would charge Miss Noether with explaining this to me. (ibid.)

Although subsequent correspondence from Hilbert or Noether on this matter is not available,¹⁴ Einstein's writing from June reveals: Your new explanation has dispelled my doubts on this point as well. [...] Now your entire fine analysis is clear to me, also with respect to the heuristics. Our results are in complete agreement. [45, Doc.224]

The mathematical details of this exchange warrant a more extensive analysis elsewhere. Here, it should be noted that the concept of energy held central importance in both Einstein's work of November 1915 and in Hilbert's contemporaneous publication, serving as a critical element of the new general theory of relativity. While both authors provided specific mathematical representation of energy, these representations carried novel meanings that neither fully understood at that time. By the end of this short correspondence, Einstein maintained that his doubts had been dispelled. However, a precise understanding of the relationship between the two energy concepts and their respective mathematical representations would not be achieved until Klein's *Second Note* of July 19, 1918, informed by Noether's theorems.

The unresolved connection between these energy concept was among the issues that later motivated Klein's investigations. It is worth emphasizing that Hilbert, recognizing Noether's mathematical brilliance, had tasked her with exploring the relationship between Einstein's and his own representations of the concept of energy within the framework of generally covariant theory. On the one hand, this assignment reflected the hierarchical academic structure of the time, in which professors delegated complex mathematical subtasks to their assistants. On the other hand, it underscored Hilbert's confidence in Noether's exceptional competence as a mathematician.

¹⁴An indirect source providing insight into a non-extant manuscript by Noether about the comparison between the energy expressions of Hilbert and Einstein is discussed by [36].

4 The Collaboration Between Emmy Noether and Felix Klein

The sources indicate that initially, there was lively contact between Hilbert and Noether in the search for the field equations of gravitation and the discovery of the generally covariant theory of gravitation. But it was Felix Klein who continually sought out the conversation with Emmy Noether during the war years and engaged with her. An important source are Klein's notes, excerpts and correspondence from these years, which have been preserved primarily in the two extensive files in his Göttingen estate mentioned above.¹⁵

4.1 The Archival Sources

These notes document a process of reflection by Klein that began in the context of a lecture series that developed its own dynamic and in 1918 led to Klein's publications already mentioned. These files contain a diversity of material, mainly lecture manuscripts, but also excerpts and notes on literature studied, notes on Klein's conversations with various of his colleagues and students, as well as various correspondence from this period on this complex of issues, for example with Walter Baade (1893–1960), Constantin Caratheodory, Friedrich Engel (1861–1941), Carl Runge (1856–1927), Arnold Sommerfeld, Hermann Vermeil (1889–1959), and others.¹⁶ Most of the documents were dated by Klein, but they are not arranged chronologically.

Since Noether plays a prominent role in Klein's notes, these documents are a rich source for the reconstruction of the mathematical discussions in connection with the development of Noether's work on "Invariant Variational Problems". As far as Emmy Noether is directly concerned, the folders contain five letters from Noether to Klein.¹⁷ as well as two manuscripts. One of Noether's manuscripts (22A, 56) is dated August 3, 1916 (22A, 56) and entitled "Differential Invariants." It contains a preliminary stage of her work on "Invariants of Arbitrary Differential Expressions" [31], which Klein submitted to the Göttingen Academy for publication on January 25, 1918. The other manuscript (22B, 138) is dated July 23, 1918, and entitled "Invariant Variational Problems." It contains the formulation of Noether's two theorems together with a sketch of the proofs on a single sheet. This was probably used by Klein to announce her paper [30] for publication in the *Nachrichten* on July 23, 1918.

In addition to these documents from Noether's hand (with Klein's marginal notes), her name is mentioned over 30 times between October 1916 and November 1918 on dated Klein manuscript pages. Thus, they reflect roughly the same number of face-to-face conversations between Klein and Noether during this period. Apart from one longer period between January and May 1917, during which Noether visited

¹⁵ SUB Cod.Ms.Klein 22A, 22B. In the following these archival folders will be cited as 22A and 22B. In my review and analysis of these notes, I have also benefited from transcripts made of some of the documents about 20 years ago by Virginia Dippel as part of a project on these notes that Michel Janssen and I began at the time but did not pursue further then.

¹⁶ Einstein's letters to Klein from these years are contained in an extra folder SUB Cod.Ms.Klein 22B, Einstein. The correspondence between Einstein and Klein is published in [45].

¹⁷ The letters are dated February 11, 15, 23, March 12 and September 16, 1918 (22B, 4–9). The letters of February 15 and March 12 have been published in [23, App.I,II].

her father in Erlangen while Klein retreated for some time to a sanatorium in nearby Hedemünden [59, p. 301], as well as a gap from August to October 1917, the mentions of Noether document regular interactions over the entire period. In fact, Klein and Noether must have consistently met at intervals of a few days or weeks for in-depth conversations, per evidence of these notes.

Although Klein's notes offer a wealth of information, they should be approached with a degree of critical caution. The manuscripts were evidently compiled by Klein himself, possibly with an eye toward historical documentation that would include his own role in the events and discussions.¹⁸ Klein, acutely aware of questions of academic priority, often gave extensive judgments on the precedence and quality of other works. Notably, in notes for his lecture to the *Göttingen Mathematical Society* on October 30, 1917, he states: "Presenting the personal side of the matter is also important to me." (22B, 40) It remains uncertain whether all dates were recorded at the time of occurrence or retrospectively.¹⁹ Regardless, the vast majority of these notes were drafted by Klein directly for his own reference. As such, they yield an exceptionally candid and authentic insight into the intellectual exchanges at the *University of Göttingen* during this period.²⁰

4.2 Klein's Lectures on the History of Mathematics

The starting point for the investigations documented in these folders were Klein's lectures on the history of mathematics. In the winter semester of 1914/15, Felix Klein began a multi-part series of lectures on this subject.²¹ In the summer semester of 1916, he announced: "conclusion of the lectures on the development of mathematics in the 19th century", scheduled as always for "Wednesday 11-1 o'clock, privatissime and free". In the winter semester 1916/17 Klein announced: "lectures on selected parts of modern mathematics", but already in the summer of 1917, a record of "resumption of the lectures on the modern development of mathematics" appears. This lecture cycle continued until 1919. Klein's notes were only published posthumously in 1926 and 1927. The preface by the editors states that Klein wrote them "during the first years of the war in his apartment in front of a small circle of people."²²

¹⁸A few pages at the end of folder 21N show that the various versions of the lecture notes and Klein's associated notes were separated by later editors.

¹⁹For example, 22A, 61, was obviously dated only later by Klein to "mid 1917".

²⁰Cf. also Tollmien [59, p. 248].

²¹The course catalog from the winter semester of 1914/15 informs that Klein "does not teach" ("liest nicht"), but the first folder with notes on the subject in SUB Cod.Ms.F.Klein 21 H is dated winter semester 1914/15. The other folders, SUB Cod.Ms.F.Klein 21J–21P, contain notes on further lectures of this cycle up to the winter semester 1916/17, with lists of participants. Noether is also mentioned occasionally in these notes. The original motivation for these lectures was Klein's plan to write a contribution on the history of mathematics in the multi-volume work on "Kultur der Gegenwart," a project which he had helped to initiate himself [50]. Klein's contribution did not appear there, but Courant and Neugebauer published his two-volume *Lectures on the Development of Mathematics in the 19th Century* based on the papers and notes that he had left behind [18, 19]; see also [51, 409–413, 455–458].

²²[18, p. V] In fact, there are also indications in the notes that the lectures took place in his "apartment", for example on April 26, 1918 (22B, 82v) or on September 3, 1918 (22B, 122).

While Emmy Noether is listed as an auditor for the winter semester 1915/16 and the summer semester 1916 (21J, Bl.63; 21K, Bl.2), her regular participation in later lectures is not known as it is not documented in the participant lists. Consequently, it remains unclear whether Emmy Noether continued to attend these lectures or if Felix Klein rather viewed her as a colleague with whom he occasionally discussed the direction of his lecture.

The underlying motivation behind Klein's lecture series on the development of mathematics in the 19th century, especially in its later parts, was his desire to bring his old "Erlangen program" to bear on the interpretation and arrangement of ideas in modern physics. While working on this overarching program, the emergence of the general theory of relativity in the fall of 1915 brought about a profound conceptual innovation in theoretical physics. What made this development particularly interesting for Klein was that at the core of this development was the invariance or covariance of central concepts in relation to a group of coordinate transformations that had not previously played a central role in physics. The general theory of relativity formulated its laws covariantly with respect to the group of arbitrary coordinate transformations, i.e. general diffeomorphisms, referred to in Klein's notes as G_∞ . This presented a challenge to the central tenet of the "Erlangen program", which aimed to classify the various geometric systems based on their associated transformation groups.

The debate with Einstein, who had not visited Göttingen since July 1915 due to wartime constraints, and his general theory of relativity were constantly in the background of Klein's interest as reflected in his notes. As early as mid-July 1916, for example, Klein wrote:

We now turn to the new work from Einstein to Hilbert. My aim is to combine the ideas of the old mechanics, inertia and gravitation, with modern electrodynamics in a new way. No contradiction to the theory of relativity of the Lorentz group, but a further development of it.

The relevant literature implied by Klein at the time included Einstein's works since 1911, Hilbert's *First Communication*, Mies' three works on the foundation of a theory of matter, and Schwarzschild's two works [46, 47] on spherically symmetric, static solutions of the relativistic field equations of gravitation. Klein explains: The difficulty is again that each author creates his own designations, taking into account the suggestions he encounters by chance. I will frame my presentation in continuity with the line of thought set out so far and hopefully succeed to bring you to a full understanding of the main ideas.

He finishes the course off by remarking: This concludes this semester's lectures, which ultimately took a completely different turn than I had initially intended. I could not resist the temptation to show how the essential mathematical thoughts later assert validity and from there, receive the most stimulating impulses. This is entirely in the spirit of this lecture and probably corresponds also with your goals. But of course—because it is necessary to go into details—it meant a

major shift in the program of the entire lecture course. (SUB Cod.Ms. Klein 21L, p. 103)

These comments on Klein's lectures on the history of mathematics from the summer of 1916, crafted in response to the emergence of the general theory of relativity, highlight a clear shift in his interests. This time in point determined his research focus over the next two years.

Parallel to Klein's lectures on the history of mathematics and Hilbert's lectures on the theory of relativity, Noether delivered her own lectures on invariant theory in the winter semester of 1916/17. In that same term, Klein revisited and updated his summer lectures. On October 19, 1916, he noted a reminder:

The lecture I would like to give on my Erlangen program in the winter semester will have to deal with the difficulty that the audience has no knowledge at all of the invariant theory of linear substitutions, which is ultimately the main point I have to make there. Miss Noether? (22A, 34)

A subsequent entry dated November 10, 1916, explicitly refers to Noether's parallel lectures: How did Miss N.'s own presentations go? (Audience) Mutual agreement on terminology and delimitation of subject matter. (22A, 54)

Meetings between Klein and Noether followed, including on November 16 (22A, 52) and on December 19, 1916 (22A, 50). In the latter meeting we find, again with reference to "Miss Noether", the annotation: Variational processes of invariants produce invariants, which is to be carried through exactly.²³

Klein surrounded this statement with a curly bracket and annotated it with: "Lecture by Miss Noether."

One of the mathematical problems that Klein sought to understand more deeply during this period was the derivation, justification and significance of the Riemann curvature tensor. With its contractions, the so-called Riccitenor and the Riemann scalar, this object played a decisive role in the formulation of the general theory of relativity for both Einstein and for Hilbert, albeit in different ways. For Klein, this problem spanned multiple areas of mathematics, requiring an approach from different angles.

[7] identified the Riemann tensor as the central point of departure for a relativistic gravitation theory:

The problem of constructing the differential equations of a gravitational field [...] draws one's attention to the *differential invariants* and *differential covariants* of the quadratic differential form:

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu.$$

²³ "Variationsprozesse von Invarianten geben Invarianten, was genau durchzuführen ist."

In the sense of our general vector calculus, the theory of these differential covariants leads to the *differential tensors* that are given with a gravitational field. The complete system of these differential tensors (with respect to arbitrary transformations) goes back to a covariant differential tensor of fourth rank found by Riemann and, independently of him, by Christoffel, which we shall call the *Riemann differential tensor* [...] [7, 35]

In later works, Einstein usually referred to the tensor as the “Riemann-Christoffel tensor” (e.g. [5, p. 1053]). Klein systematically pursued the origins of this object in line with his narrative of a history of mathematics. “After Christmas” in 1916, Klein and Noether discussed the “real production of invariants according to Christoffel and Ricci” (22A, 50). Further conversations took place on January 22 and 23, 1917. Klein’s notes on this subject include the remark “For Miss Noether”: Now pure mathematics again, liberated from work of man.²⁴ (22A, 48)

Beginning in January 1917, their dialogue paused while Noether spent some time in Erlangen and Klein retreated to Hedemünden.

Concurrently, during the same winter semester of 1916/17, Hilbert conducted the second part of his lecture course on “Foundations of Physics”, which focussed primarily on the general theory of relativity [43, 162–308]. He also gave talks at the *Göttingen Mathematical Society* on “The Principle of Causality” on November 21, and on “Non-Euclidean Geometry and the New Theory of Gravitation” on January 23. While no direct references to collaborations between Hilbert and Noether appear in the lecture materials, Hilbert ended the course with the cryptic statement:

The curious theorem applies that the number of equations flowing from Hamilton’s principle always corresponds to the number of unknown functions, except in the case that the integral is over a general invariant. [43, S.159]

The lack of evidence of interaction between Hilbert and Noether does not imply that there was no further cooperation. In fact, Hilbert was offered a professorship in Berlin at the end of 1916, but after negotiations in Berlin on January 15, he signed an agreement to remain in Göttingen. It stipulated, among other things: Furthermore, it is taken into prospect to grant Miss Nöther a remuneration after the war in order to keep her for the Göttingen seminary. (quoted after [42, p. 191])

For the summer semester of 1917, Klein planned to resume the discussion. On May 31, he made a note suggesting an “Essay on the Invariant-Theoretic Construction by Means of Variational Approaches”²⁵ and to design the “constitution of my new lecture.” He also jotted down: How do I reach Miss Noether? What is she doing? (22A, 5)

²⁴ “Nun wieder reine Mathematik, von Menschenwerk befreit.”

²⁵ “Aufsatzes über den invar.th.Aufbau durch Variationsansätze.”

On July 9, Klein and Noether met to discuss the generation of differential invariants by series expansion in Riemannian normal coordinates. This meeting also revolved around “Hilbert’s Four Theorem” (“Hilbert’s Vierersatz”) and a comparison of the two energy expressions. Klein recorded: The energy tensor is not yet invariant with Einstein [...] and becomes so with Hilbert only by means of addition of an auxiliary vector. (22A58)

Their exchange continued three days later, on July 12. Interestingly, on this day, the subject of debate was an “explanation of the old note of 3/8 16.” (22A, 58v) It seems that Emmy Noether had already played an important role much earlier, as she was obviously well acquainted with covariant derivatives from the point of view of invariant theory. The “old note” is the page already mentioned, written in Noether’s hand, on “differential invariants” (22A, 56), with a marginal remark by Klein: “Notes by Miss Noether 3/8 16.” Here Noether outlines an invariant-theoretical method for generating “all differential invariants of the fundamental form, and any other forms and their derivatives.” The equation of the shortest line and a variation over the Riemann tensor are discussed as examples.

However, Klein’s approach to analyzing the Riemann tensor differed significantly from Noether’s approach. Klein commissioned his assistant at the time, Hermann Vermeil, to carry out detailed calculations on his behalf to derive the Riemann tensor. Vermeil regularly communicated his results to Klein in writing, and at the end of October, Klein finally submitted a “Note on the Mean Curvature of an n -dimensional Riemannian Manifold” [61] to the *Göttingen Academy*. Shortly after, Klein delivered a series of lectures to the *Göttingen Mathematical Society* “on Riemann’s general definition of his measure of curvature of a quadratic differential form” (22B, 49). The talks took place on October 30, November 6, and December 4, 1917.²⁶ Interspersed in this series was a report by Runge on Hessenberg’s recently published “Vectorial Foundation of Differential Geometry” [11] (22B, 61). In his lectures (22B, 40–49), Klein gave a very detailed account of how he understood the invariant-theoretical justification of differential invariants of a quadratic form. His notes on this issue show only one mention of Noether, on December 4, “Miss Noether. Only make appointment” (“Frl. Noether. Nur verabreden.”) (22B, 46). In his lecture, we find the demarcation of her approach to that of Vermeil:

Vermeil proved this at my suggestion by applying the partial differential equations for orthogonal invariants specifically to linear invariants.

Miss N. does it with symbolism following Weitzenboeck. (22B, 48v)

These efforts laid the groundwork for three lectures in January 1918, constituting the actual prelude to the development of Klein’s and Noether’s work.

²⁶According to the *Mitteilungen und Nachrichten in den Jahresberichten der Deutschen Mathematiker-Vereinigung* (JDMV-MN), vol. 27, p. 28, Klein lectured on 10/30 and 11/6 about “the Riemann curvature measure and related topics”, but on 11/6 “in particular [about] geometric interpretations”, and on 12/4 “about Herglotz, interpretation of Hilbert’s curvature and the associated gravitational tensor”. Short summaries of Klein’s lectures can also be found there.

On January 15, Emmy Noether presented a lecture “on Invariants of Arbitrary Differential Expressions”. The summary of this lecture in the *Mitteilungen und Nachrichten* of the DMV reads:

The author succeeds in proving the theorem that Christoffel established in 1869 for quadratic differential forms: namely that the invariants which such an expression possesses with respect to any analytic transformations $x_i = \varphi_i(y_1 \dots y_n)$ can be largely generalized to the invariants which a certain infinite series of associated differential forms possesses with respect to any linear transformations $dx_i = \sum c_{in} dy_n$ (will appear in the *Göttinger Nachrichten*). (JDMV-MN 27, p. 28)

On January 25, Klein submitted a paper with the same title as Noether’s work [31] for publication to the Göttingen Academy on January 25.²⁷ Three days earlier, however, he delivered a lecture at the *Göttingen Mathematical Society*, titled “on Hilbert’s First Note on the Foundations of Physics”. The discussion following his lecture continued a week later, as noted in the *Mitteilungen und Nachrichten*, “29. January 1918. Hilbert, continuation of the discussion on the foundations of physics”. The summary provided reads: More detailed explanations of the issues involved can be found in a correspondence between Klein and Hilbert, which will soon be published in the final issue of *Göttinger Nachrichten* for 1917. Suffice it here to emphasize the following result: The “conservation laws” of classical mechanics (the momentum-energy laws), which are relevant to the mechanics of continua, are already contained in the field equations of the new theory inaugurated by Einstein; they thus lose their independent significance. (JDMV-MN 27, p. 28)

Indeed, at the Göttingen Academy meeting on January 25, Klein presented not only Noether’s paper but also his own publication titled “On Hilbert’s First Note on the Foundations of Physics” [16].

This publication by Klein from January 25 has a peculiar format. It consists of three parts, each titled as excerpts from letters: “I. From a letter from F. Klein to D. Hilbert...II. From the reply from D. Hilbert...III. From another letter from F. Klein” [16, pp. 469, 477, 481]. It was not unusual for contributions to the *Göttinger Nachrichten* to be framed as letter excerpts; similar formats appear in Engel’s works [9, 10], and Vermeil’s [61], such as “From a letter to F. Klein.” In this case, however, the result was that Hilbert himself did not appear as an official author, despite being a de facto co-author.

Noether’s role in this publication is particularly noteworthy. Toward the end of his first part, Klein wrote:

²⁷ Einstein wrote to Hilbert about this work: “Yesterday I received a very interesting paper by Ms. Noether about the generation of invariants. It impresses me that these things can be surveyed from such a general point of view. It would not have harmed the Göttingen soldiers [“Göttinger Feldgraue”] to have been sent to Miss Noether for schooling. She seems to know her trade well!” [45, Doc.548]. About the meaning and translation of the German expression “Göttinger Feldgraue”, see [48].

I have to make a significant intervention here. You know that Ms. Noether continues to advise me in my work and that I have actually only penetrated this subject matter through her. [16, p.476]

Klein further specifies: When I recently spoke to Ms. Noether about my result concerning your energy vector, she was able to tell me that she had already derived the same from the developments of your note (i.e. not from the simplified calculations of my No. 4) a year ago and had laid it down in a manuscript at that time (which I then inspected); she had only not asserted it with such determination as I recently did in the Mathematical Society (January 22) (ibid.)

Noether's manuscript which Klein referred to has not survived. Based on Klein's dating of "before the end of the year," it would have been composed in late 1916 or early 1917. David Rowe later discovered a document in Rudolf Humm's estate that might be an incomplete copy of this manuscript [36].

In his answer, Hilbert also acknowledged Noether's contributions, dating her involvement with this question even earlier:

I completely agree with your comments on the energy theorem: Emmy Noether, whose help I called upon more than a year ago to clarify such analytical questions concerning my energy theorem, found at that time that the energy components I had set up—just like Einstein's—can be transformed formally by means of the Lagrangian differential equations [...] into expressions whose divergence vanishes *identically*, i.e. disappears without using the Lagrangian equations [...]. [16, p.477]

Hilbert contrasted this own observation with classical mechanics and electrodynamics, making the following assertion: ...that for *general* relativity, i.e. in the case of *general* invariance of the Hamiltonian function, energy equations, which in your sense correspond to the energy equations of the orthogonal invariant theories, do not exist at all; indeed, I would even venture to call this circumstance a characteristic feature of general relativity. For my assertion, a mathematical proof should be possible. [16, p.477]

Klein referenced Noether's findings in his own unpublished notes for the January 22 lecture, yet without mentioning her manuscript. He prepared to include: The occasion for today's lecture is that I believe I can shorten the obscure intermediate calculations. But there remain scruples concerning the so-called conservation laws. (22B, 50)

The substantive technical issues underlying this statement warrant an in-depth analysis elsewhere. Nonetheless, it is important to emphasize the culmination of discussion in January 1918: three lectures at the *Mathematische Gesellschaft* delivered by Noether, Klein, and Hilbert; two articles in the *Göttinger Nachrichten*, one by Noether, one by Klein, spaced a week apart. Moreover, the latter incorporates contributions by Hilbert in the form of letter excerpts. Then, there is the controversial

debate centered on a question for which Noether had written a manuscript at least a year earlier, but which was neither fully discussed nor preserved. Lastly, there is Hilbert's assertion, spurred by Noether's investigations, for which the mathematical proof remained outstanding.

Despite all the support for Emmy Noether received from both Klein and Hilbert, Tollmien [56, p.180] observed that, by contemporary standards, Noether could, and arguably should have been credited as co-author of Klein's 25 January note. While it is true that Klein had already submitted another paper by Noether to the academy for publication on the same day, this only raises the additional question of whether that publication might have been expedited earlier, as Klein had done for Vermeil's paper in the fall of 1917. And as far as his own manuscript of January 25, 1918, is concerned, Klein and Hilbert opted for the unusual format of letter excerpts, which resulted in Klein being the sole credited author of an investigation that critiqued Hilbert's earlier work while, at the same time, giving Hilbert a platform to respond. A joint publication with Hilbert may have been impeded not only by character differences and prevailing conventions but also by Klein's criticism of Hilbert's overly complicated presentation. Noether, on the other hand, collaborated closely with both Klein and Hilbert, earning recognition for her findings from both. Yet, it was precisely her characteristically neutral stance that led her to be caught between two fronts.

Klein's later article in July introduced itself as a direct "continuation of the investigations which I presented to the *Gesellschaft der Wissenschaften* on January 25 of this year." [17, p.171] However, this statement obscured the fact that Klein's trajectory between January and June was far from linear.

Following the debate with Hilbert, Klein probably requested from Noether the proof for the assumption that there could be no actual energy theorems in the case of general covariance. On February 15, Noether wrote to Klein:

I first considered the energy theorems in the simplest case, which is a direct generalization of $f(z_1, \dots, z_n, dz_1, \dots, dz_n)$ for the simple integral. (Noether to Klein, February 15, 1918, 22B,8)

She outlined her approach based on what she described as "Lagrange's central equation" and concluded: I hope to be able to handle the general case where tensors $g_{\mu\nu}$ are used instead of scalars z_ν in a similar way. (ibid.)

Direct proof of Hilbert's assertion seemed difficult at first. Klein appears to have proposed in a letter to Noether (which is lost) that the assertion should be proved indirectly by contradiction. About a week later, Noether responded: I have therefore concluded the non-existence of an integral analogous to the energy theorem from the formulas; this is not very satisfactory, since no inner reason can be seen; the formulas give clarification, instead of inner reason, for the fact that the ordinary energy theorem goes over into Lie's differential equations in the case of invariance; but perhaps someone else will do better later. (ibid)

Outlining her results, which already anticipated her two famous theorems, Noether concluded the letter: I want to work this out now, but it won't happen very quickly! (ibid.)

While Noether was clearly already advancing toward the completion of her renowned paper, Klein was preoccupied with other projects and, as events unfolded, diversions.

In reflecting on the energy concepts, developed by Einstein and Hilbert, Klein believed he had discovered a surprisingly simple solution. Together with his friend and colleague Carl Runge, Klein proposed what they termed the “Egg of Columbus.” Their solution involved attributing no energy contribution at all to the gravitational field. Unaware that this idea was not original, Klein and Runge prepared two communications for the *Göttinger Nachrichten* in February and March, explaining their concept. Two versions of Klein's note have survived. Within these, he analyzed Schwarzschild's solution to the field equations of general relativity and demonstrated that, on their assumption, the calculated energy aligned approximately with classical results, which only accounted for mass density and gravitational self-energy. Runge's paper has not been preserved. It reportedly argued that a mathematical “trick” could reduce the covariant energy equation to the familiar form of special relativity, even for general gravitational fields.

Klein expressed great confidence in that this solution would resolve the ongoing debate conclusively. His notes from March 5 reflect his conviction:

Finally, the matter becomes frighteningly simple and Hilbert in his answer and now Ms. N. search unnecessarily deep. (22B, 89)

However, his manuscript also contains an interlineation, possibly inserted at a later date, in which Klein admitted: I should know more elementary relativity. (ibid.)

Ultimately, the “Egg of Columbus” proved to be a premature, flawed concept. Surviving versions of Klein's notes reveal his train of thought and attempts he made to implement the idea. In the first draft, Klein identified elementary errors in his calculations, noting to himself, “full nonsense” and “also still wrong” (22B, 70, p.4/5). However, the physical and conceptual difficulties inherent in their approach presented greater obstacles. They were clarified in correspondence with Albert Einstein. This exchange eventually led to Klein and Runge withdrawing from publication. On May 18, they informed Einstein: Based on our recent correspondence, Runge and I have deferred our planned publications and intend to return to them, should the occasion arise, once we have a complete overview of the currently available literature. [45, Doc. 540]

While Einstein's objections likely had a decisive part in exposing the untenability of their proposal, Noether also played a role. Drawing on her own understanding of the variational problem, she criticized the proposal early on. Writing to Klein on March 12, she expressed: Thank you very much for sending me your note, and for your communication about Runge's Egg of Columbus. I

do not entirely agree with it; it fails precisely in the simplest case, that of first-order homogeneity; [...] (Noether to Klein, 22B, 7)

On March 23, Klein made a note of a conversation with Runge: Notes Ms. Noether. Ms. Noether and Hilbert's assertion (22B, 62)

A week later, on March 30, another note read: Orientation of the new works by Ms. Noether and Vermeil (22B, 65)

The failed “Egg of Columbus” was not the only detour in Klein's continuation of his investigations. He also became interested in cosmological discussions involving Einstein, de Sitter, and others [45, editorial note, pp. 351–357]. These exchanges seemed to inspire him to consider publishing on this topic himself.

In addition to these academic pursuits, Klein was involved in organizational activities, in particular, 20th anniversary celebration of the *Göttinger Vereinigung* and hosting a multi-day visit by Max Planck to Göttingen, referred to as the “Planck Week”.

Klein also maintained his commitment to an extensive lecture cycle on the development of mathematics. By the end of April, he began his lectures for the summer semester. In his preparatory notes, he wrote:

Everything comes out somewhat differently than in the other representations, because the ideas of the Erlangen program and the connections of the general historical development on the mathematical side hover over everything. (22B, 82v)

Reflecting on carrying out his project, he added: But I lack the specific physical knowledge and, on the other hand, also something on the mathematical side: the real meaning of the algorithms.

Therefore, not a one-sided lecture, but a collaboration with you: Individual discussions, individual lectures in my apartment, and lectures in math Society. All the more so as I am in the stage of depression from which you must lift me out of it. (22B, 82v)

The phase of depression may have been tied to the failure of the “Egg of Columbus”. Klein also detailed his plans for “individual reviews”, where Noether's name appeared again: Pure Math.[thematics] Vermeil (Ms. Noether).

Around this time, Klein appeared to shift focus back to questions surrounding the energy concept in general relativity, perhaps encouraged by Einstein's presentation to the *Berlin Academy* on May 16, 1918 [6], a copy of which Klein received at the end of May (Einstein to Klein, May 28, 1918 [45, Doc. 549]). Klein's renewed attention to this topic seemed to come from the consistent treatment of all the boundary integrals in the variational problem. Noether, it seems, had pursued this insight on a mathematical level earlier, as suggested

by Klein's note dated on June 7, 1918: Miss N. claims that my way of varying (Hilbert's note) is wrong in itself, but that it can be justified afterwards and is then very useful. (22B, 111)

There appears to have been a period where contact between Klein and Noether cooled, possibly linked to Klein's depressive stage. By the summer of 1918, however, their mutual consultations resumed and peaked in a series of successes.

The *Göttingen Academy's* "Register of Meetings Held in 1918/19 and the Scientific Papers Presented at them" lists Felix Klein's contribution "On the Differential Laws for the Conservation of Momentum and Energy in the Einstein-Hilbert Theory" presented at the "regular meeting" of July 12, 1918. For the July 26 regular meeting, the following entry shows:

Felix Klein announces: Emmy Noether (Göttingen), Invariant Variational Problems.

Publications in the *Proceedings of the Academy* were communicated by a member of the Academy. As a result, Felix Klein was able to submit his own work independently, whereas Emmy Noether had to rely on Klein to communicate her contributions.

It is worth noting that Klein's published paper carries a slightly modified title. Hilbert's name was omitted, and the final title simply reads "... in Einstein's theory". Additionally, the published paper indicates a slightly later date, stating that it was presented "at the meeting of July 19, 1918".

Beyond the formal business meetings of the *Göttingen Academy*, presentations and discussions also took place in the *Göttingen Mathematical Society*, a less formally regulated circle.

Klein's announcement of his article at the July 12 meeting was evidently intended to assert priority. However, he most likely had not yet completed the manuscript in any publishable form at that point. On the contrary, it is quite clear that Klein had not yet fully grasped all aspects of the material as he would present them in the paper.

On July 15, Klein wrote to Einstein:

I have succeeded in discovering the organic formulation of the law for Hilbert's energy vector. [...] I hope very much to pave the way from here, also in conformance with my assumptions [...]. But with other demands on me and reduced productivity, it is all just going very slowly. [45, Doc. 588]

The following day, Klein gave a presentation at the *Göttingen Mathematical Society*, for which a detailed manuscript survives (22B, f116–118). The lecture exclusively focussed on "Hilbert's Energy Vector" with Klein demonstrating how the energy vector arises from a consistent treatment of the boundary integrals under an infinitesimal transformation of the variation integral.

At the conclusion of the presentation, Klein noted:

Incidentally, my considerations are closely related to those that Ms. N. will present tomorrow. The historical context is probably that we only came to full clarity together.

On July 17, Noether delivered her lecture, though no archival evidence seems to have survived to shed light on its content.

On July 22, Klein gave another lecture at the *Göttingen Mathematical Society*, for which a manuscript is extant (22B, 119–121). The lecture, too, centered around the discussion of “Hilbert’s Energy Vector”. Klein noted for the lecture:

Continuation of the reflections I gave exactly 1/2 year ago on Hilbert’s First Note on the Foundations of Physics. Summarize everything so that it is understandable. Good agreement and in any case *Burgfrieden* until after the end of the lecture.

The ambiguous phrase about a “castle truce” (“*Burgfrieden*”) invites speculation. Klein may have anticipated or encountered opposition, though he refrained from naming individuals. One may speculate that he might be referring not only to his differences with Hilbert, but also to tensions with Noether. In any case, the phrase itself is curious, as it alludes to the political situation at the beginning of the First World War, where “*Burgfrieden*” was the metaphor for Kaiser Wilhelm’s declaration “I know no more parties”, made in connection with the Social Democrats’ agreement to the war credits. If Klein’s term refers to his differences with Noether, it would carry a certain irony. Despite his progressive and international outlook in academia, Klein was one of the 93 signatories, and the only mathematician, of the infamous *Aufruf an die Kulturwelt*. Although Klein signed the document in ignorance of the actual wording of the pamphlet, he never fully or publicly revoked his signature [55, 51, pp. 451–454]. Noether, by contrast, joined the USPD, the Independent Social Democratic Party, a left-leaning anti-war split-off of the Social Democratic Party, in 1919 [1, 24, 40, 51, 60].

Regardless of these dynamics, Klein’s July 22 lecture was likely delivered to a larger audience. His lecture notes indicate meticulous preparation, including the use of slides to project key formulas instead of writing them on the blackboard. His assistant Walter Baade (1893–1960) helped to create the slides.

This lecture specifically dealt with the derivation of Hilbert’s energy vector from a consistent invariant-theoretical treatment of the variational principle. At the end of his notes, Klein jotted down (see also Fig. 1):

Finally a relationship with Miss Noether again. (22B, 121v)

He added Before everything was completely clear, we had a few arguments. In the end, it turned out that the more general theorems that she will present tomorrow at the Mathematical Society correspond exactly to the essence of my approach. We will therefore best postpone until tomorrow the question of how my approach can be applied to ordinary mechanics.

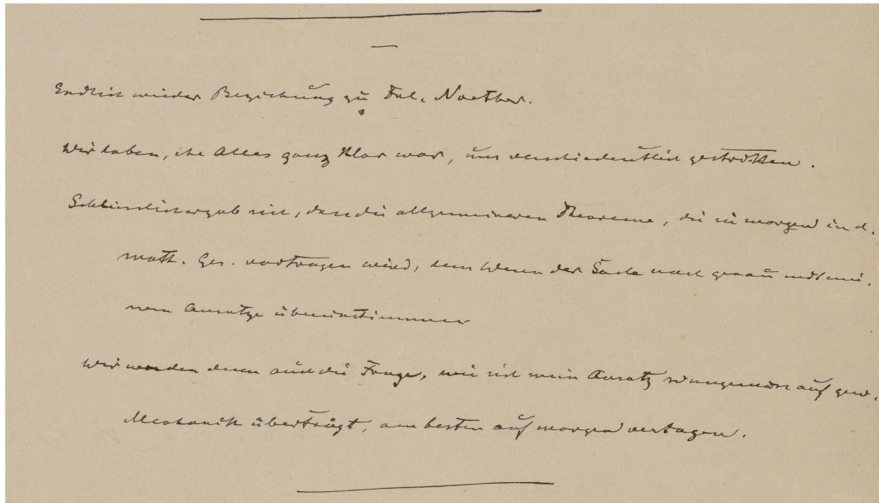


Fig. 1 Clipping from Felix Klein’s notes (SUB Cod.Ms.Klein 22B, 121v). It says: “Endlich wieder Beziehung zu Frl. Noether. // Wir haben, ehe alles ganz klar war, uns verschiedentlich gestritten. Schliesslich ergab sich, dass die allgemeineren Theoreme, die sie morgen in d. math. Ges. vortragen wird, dem Wesen der Sache nach genau mit meinem Ansatz übereinstimmen. // Wir werden denn auch die Frage, wie sich mein Ansatz sinngemäss auf gew. Mechanik überträgt, am besten auf morgen vertagen.”

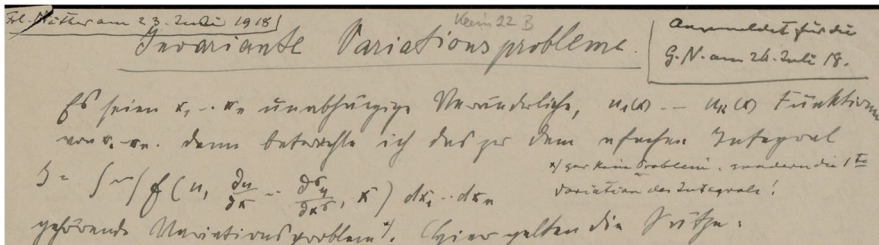


Fig. 2 Top of a two-page document containing the announcement of Emmy Noether’s two theorems on “Invariant Variationsprobleme” in her own German script (SUB Cod.Ms.Klein 22B, 138). In the top left corner, Klein wrote “Noether am 23. Juli 1918,” and in the top right corner, he wrote “Angemeldet für die G.N. am 26. Juli 1918”

Unfortunately, details about Noether’s lecture at the *Göttingen Mathematical Society* from July 23 are not known.

However, in Klein’s folders, with a dating note in Klein’s hand, “Miss Noether on July 23, 1918”, the summary of the results of her later published work can be found in Noether’s handwriting. The document is written in Noether’s characteristic German script, but on the top right-hand side of the page, also in Klein’s hand, is the additional date, “Announced for the G.N. on Juli 26, 1918,” see also Fig. 2.

This note matches the dateline on the version published in the *Göttinger Nachrichten*:

Presented by F. Klein in the Meeting of July 26, 1918.

The title “Invariant Variational Problems” already matches that of the published paper and the summary explicitly formulated the two theorems, including their inversions, while also outlining their proof.

As stressed previously, Klein’s article provides evidence that his work emerged from a collaborative discussion process. A page in Klein’s folders (22B, 132) reveals that Klein and Noether met again on August 1, 1918, to discuss two questions. One was about the “10 integrals of the two-body problem” and the other about the “energy vector for arbitrary field equations.”

Further notes by Klein document his extensive study of the physical and mathematical literature extensively even after registering his article for publication. He also discussed details of the problem with his students and colleagues though it is not documented whether Noether was present during these discussions.

Finally, on September 16, 1918, Noether wrote to Klein (22B, 9) informing him that she would send him her note within a week. The delay, she explained, was due to the need to work through some points in greater detail. In particular, she noted that, during the final revision, she had returned to her original approach, which was based on the invariance of the integral of variation. This differed from the interim approach, which had sought to ground the proof on the invariance of the first variation. In addition, she pursued the question of the invariance of the first integrals “a little more closely”. She concluded her letter by expressing that she was looking forward to receiving the page proofs for Klein’s note.

5 Concluding Remarks

On December 27, 1918 Einstein wrote to Klein:

What prompts me to write today, though, is a different matter. Upon receiving the new paper by Miss Noether, I again feel it is a great injustice that she be denied the *venia legendi*. [45, Doc. 677]

Despite their scientific prestige and reputation, Klein and Hilbert had been unable to help secure Emmy Noether’s habilitation. This only became possible after the war and the fall of the German monarchy. A few days after Einstein’s letter, Klein took the initiative and wrote to the new Ministry of Culture [52, pp. 541–542]. This time the efforts succeeded, and Emmy Noether delivered her formal habilitation lecture on June 4, 1919. As her qualifying thesis, she submitted her article, “Invariant Variational Problems”.

This work would bring her lasting legacy. It set the mathematical basis of a deep inner connection between symmetries and conservation laws. Her work marked the culmination of a long tradition of studying conservation laws in mechanics and

electrodynamics, while simultaneously laying a mathematical foundation for further development. However, the reception of Noether's theorems in physics was by no means straightforward or immediate. The broader significance of her first theorem in quantum mechanics and quantum field theory, in particular, was only fully recognized and fully applied in physics much later.

Unfortunately, we have little documented insight into Noether's own experiences while developing this work in Göttingen during the First World War. Her relevant writings, correspondence, and manuscripts have largely been lost. The few preserved letters referencing this work were retrieved from Felix Klein's extensive records, together with his own marginal comments, reflections, and assessments. Klein's perspective may have been biased by his dual role of an overbearing mentor and a collaborator, striving for an equal partnership. Yet, it is only through the lens of Klein's notes, that we obtain impressions of the intellectual and social environment in which Emmy Noether's cutting-edge work was developed.

Tollmien's [58, p. 43] argument, suggesting that Hilbert and Klein were pursuing "their own selfish interest" by inviting Noether to Göttingen, bears plausibility. The two mathematicians may have been motivated by recruiting support from an expert in invariant theory for their own long-standing work on the theory of relativity. Indeed many references in the archival documents indicate that Noether's involvement in addressing the new mathematical challenges raised by the general theory of relativity was in response to suggestions or specific requests from Hilbert or Klein.

Characteristic for Noether's mathematical disposition, she immediately and consistently abstracted these concrete problem from their physical applications. This approach, generalizing the problem as far as possible into purely mathematical terms, led to significant insights early on. Hilbert and Klein only realized similar insights after extended discussions whereas Emmy Noether was not only neutral towards the physical application context, but also towards competing interpretations by the Göttingen mathematicians. At the same time, the documents imply that this neutrality may have been grounded in Noether's own mathematical research interests. Her style of thinking, as Mechthild Koreuber [20] aptly calls it, evolved independently of the discussion surrounding the theory of relativity. It is perhaps not merely intellectual generosity when Noether wrote to Klein in one of her few surviving letters of her own that "perhaps someone else will do better later." In that same letter, her mathematical mindset is also reflected by her frustration that the formulas only produced "clarification, instead of inner reason."

When developments eventually converged in the summer of 1918 resulting in simultaneous publication of the papers by Klein and Noether, their differing perspectives also came to light. Without getting involved in Klein's line of argumentation, Emmy Noether focussed on the purely mathematical point of view. In contrast to Klein's contribution to a debate, which sought to resolve Einstein's and Hilbert's conflicting views on the concept of energy, Noether wrote her own paper proving two mathematical theorems including their inversions. Hence, her work established a mathematical basis relevant in many fields of physics, not only for this particular application related to the concept of energy.

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