True Straight-line Linkages Having a Rectilinear Translating Bar

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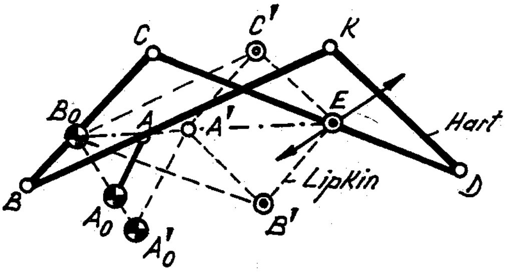
Summary - Eight- and ten-bar straight-line mechanisms with a rectilinear translating bar, have been derived from two basic types, namely from Watt's - and from Stephenson's type of Hart's true straight-line mechanism(s). Though the generalization of Watt's type, representing Hart's inversor, was easily done, the other one, namely Stephenson's type, representing Hart's 2nd straight-line mechanism, required a new design circumventing Burmester's design of focal linkages. In the end spectacular results emerged based on a new but easy going design of these true straight-line linkages. Because of the simplification obtained here, the applicability of these "lifting devices" now comes within easy reach of the designer.

1. Introduction

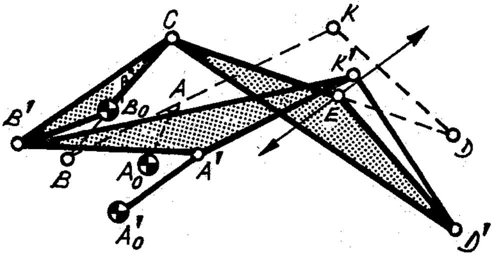
Planar linkage mechanisms producing an exact straight-line are very rare. The more so when the elementary pairs contained in the mechanism, are restricted to revolute pairs (i.e.turning-joints) only. Four-bar linkages for instance, do not produce a straight-line at all. But, **six**-bar linkages may produce them, provided the dimensions are specifically chosen. It appears that principally only *two* solutions exist. Both have been found by H.**Hart**, ref.[1], and are known as respectively, the *first* and the *second* straight-line mechanism of Hart. The first one is a six-bar of **Watt**'s type, whereas the second one may be recognized as a six-bar of type **Stephenson**, ref.[2].
The first one represents the *inversor* mechanism (figure **1**), while the 2nd one represents a special case of **Kempe**'s *focal* mechanism (figures **2** and **9**), ref.[3,4].
Hart's inversor has further been generalized, which lead to the so-called *quadruplane inversor* of **Sylvester** and **Kempe** (figure **3**). The result though, may still be recognized as a six-bar linkage mechanism of Watt's form.(ref.[5])
The generalization of the focal type, however, does lead to an *eight*-bar but then containing a rectilinear translating bar. (figure **4**, see also figure 10 of ref.[4])

A. J. Lenarčič and B. B. Ravani (eds.), Advances in Robot Kinematics and Computationed Geometry, 411–420.

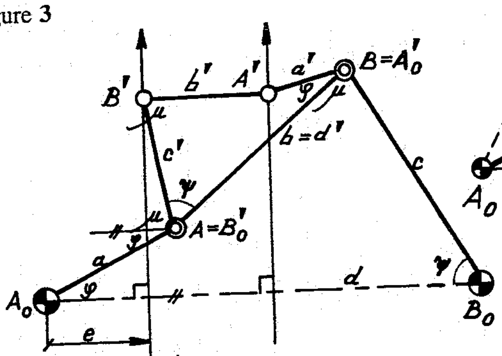
Eight-bar linkages with such a rectilinear translating bar, may also be obtained from a design based on Harts' inversor. (figure 5, see also figure 10 of ref.[6].)



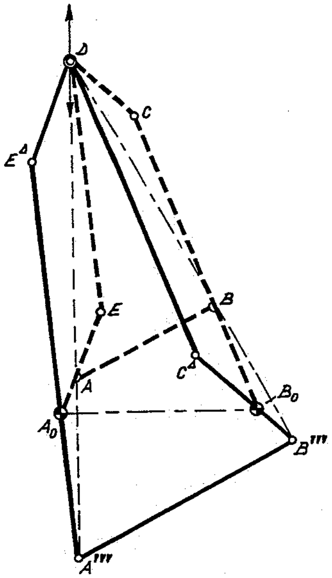
1st straight-line mechanism of Hart mechanically interconnected with the *Peaucellier-Lipkin inversor* of 1864.
Figure 1



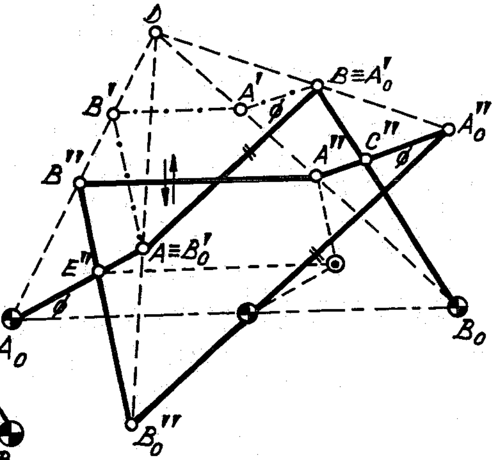
Quadruplane inversor of Sylvester and Kempe (a generalization of Hart's inversor)
Figure 3



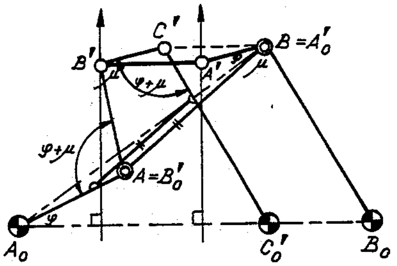
A random 4-bar and a reflected similar one, built on top Figure 7



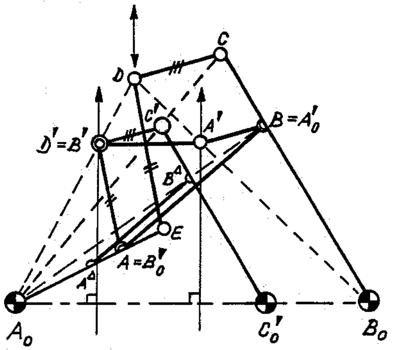
Two *curve-cognates* each representing the 2^{nd} *straight-line mechanism of Hart*(1877)
Figure 2



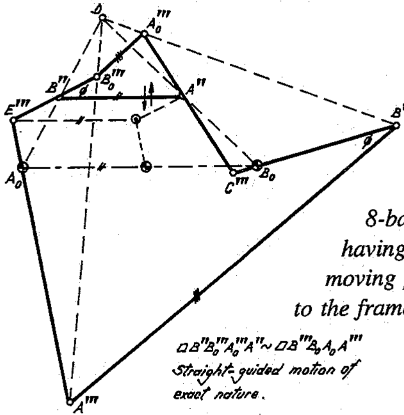
8-bar linkage mechanism with bar A''B'' moving perpendicular to the frame
Figure 4



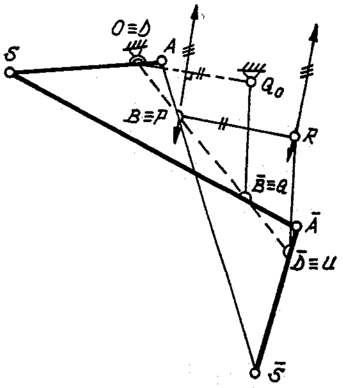
Hart's 2nd straight-line mechanism incorporated Figure **8**



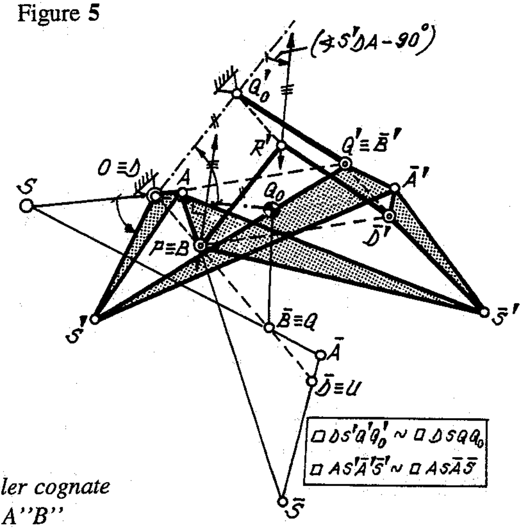
Hart's 2nd straight-line mechanism obtained by multiplication Figure **9**



8-bar coupler cognate having a bar A''B'' moving perpendicular to the frame Figure **10**



8-bar linkage mechanism with a rectilinear moving bar PR (the random *contra-parallelogram*, ASĀS̄, represents a sub-chain of the 8-bar) Figure **5**



8-bar linkage mechanism containing a bar PR' moving in an invariable but oblique direction with the frame Figure **6**

2. Eight-bar Inversor with rectilinear translating bar

In practice, only one equilateral dyad-linkage is needed to adjoin Hart's inversor in order to obtain an 8-bar with a translating link moving perpendicular to the frame, (figure **5**). The mechanism contains a contra-parallelogram and two identical dyad chains, remaining parallel during the motion.

The same procedure may be applied at the quadruplane inversor of Sylvester and Kempe.

Then, the equilateral dyad $PR'\bar{D}'$ adjoined to the 6-bar, remains parallel to the identical dyad $OQ_0'Q'$ giving a rectilinear translating bar PR' moving straight with respect to the fixed link OQ_0'. (figure **6**) The angle enclosed between the fixed link and the direction of the straight-lines, traced by points of PR' , then differs from 90^0 and equals
($\measuredangle$ S'DA - 90^0).
We conclude that this kind of generalization practically results into a change of direction of the straight motion of the translating bar. (See also figure 27 of ref.[7].)

3. Six-, eight- and ten-bar linkages derived from the focal type

Hart's 2nd straight-line mechanism, which is a six-bar of type Stephenson, is to be generalized in the way Kempe did with his focal linkage.(ref.[8] and [4])
However, it is much easier to follow a different road: Doing that, one obtains Hart's 2nd straight-line mechanism as a besides. In order to design the mechanism as successively drawn in the figures **7** and **8**,

1. one starts with a completely *random* four-bar A_0ABB_0 ,
2. then, one adjoins the chain $A_0'A'B'B_0'$ in such a way that the 4-bar $A_0'A'B'B_0'$ is *reflected similar* to the initial four-bar A_0ABB_0 , whereas
   $A_0' \equiv B$ and $B_0' \equiv A$.
3. One further adjoins the linkage-dyad B'C'B in such a way that B'C'BA' forms a linkage parallelogram.
4. One finally adjoins the bar $C'C_0'$, such that $B_0BC'C_0'$ too forms a linkage parallelogram, of which B_0C_0' represents the frame-link.

The result is a **10-bar** linkage mechanism of which the bar A'B' moves perpendicular to the frame.

Considering the stage in which the three bars C'B , C'B' and $C'C_0'$ are not yet installed and assuming that the 4-bar $A_0'A'B'B_0'$, built on top of the random 4-bar A_0ABB_0 , *remains* reflected similar with the latter, it is still easy to prove that A'B' *remains parallel* to the frame A_0B_0:

$$\sphericalangle\,(\mathbf{B'A',A_0B_0}) \equiv \sphericalangle\,AA_0B_0 - \sphericalangle\,A_0AB' + \sphericalangle\,A'B'B_0' \equiv$$

$$\equiv \sphericalangle\,AA_0B_0 - \sphericalangle\,A_0AB + \sphericalangle\,B'B_0'A_0' + \sphericalangle\,A'B'B_0' \equiv$$

$$\equiv \sphericalangle\,AA_0B_0 - \sphericalangle\,A_0AB + \sphericalangle\,A_0B_0B + \sphericalangle\,B_0BA \equiv 0^0\,, \tag{1}$$

practically using the fact that the four angles of the random four-bar are all to be indicated at the turning-joint $A \equiv B_0'$. (The sum of them being 360^0.) Thus, if we draw a horizontal passing through A, the bar $B_0'B'$ encloses the same angle μ with the horizontal as well as with the bar A'B'.

Further note that because of the reflected similarity the bar AB encloses the same angle with the fixed link as with the bar A'B'. (Actually, this remark represents the shortest proof for the parallel motion of A'B')

The reflected similarity of the four-bars may be enforced in different ways. One of them is shown in figure **8** through the adjoining of the three bars having C' as their common joint. Then, the parallelism of the bar A'B' is enforced. But other methods are allowed too. (Think, for instance of the later to be explained possibility to replace the bar C'B by a bar interconnecting A_0A and $C_0'C'$, simultaneously avoiding the appearance of linkage parallelograms.)

Notwithstanding the shown parallel motion, it still necessary to prove that a singular point of A'B' moves perpendicular to the frame. Choosing the point B', it suffices to demonstrate that the dyad-chain A_0AB' represents an eccentric crank-and-slider mechanism. Whence, it remains to prove that the *eccentricity* "*e*" of that mechanism shows to be a constant.

Clearly, the distance *e* between the fixed center A_0 and the perpendicular let down from B' on to the fixed link may be determined by projection of the dyad A_0AB' on to the frame:

$$e = a.\cos\varphi - c'.\cos\mu = a.\cos\varphi - (bc/d)\cos\mu \tag{2}$$

in which $\varphi = \sphericalangle AA_0B_0$ and $\mu = \sphericalangle B_0BA$ (3)

(thus, φ and μ are opposite angles in the initial four-bar)

Further note, that the subsequent dimensions of the reflected similar four-bar are respectively: $a' = ab/d$; $b' = b^2/d$; $c' = bc/d$ and $d' = b$, (4)

the initial dimensions being a, b, c and d respectively.

The *Rule of Cosine* twice applied in the random four-bar (abcd) on top of which the reflected similar four-bar (a'b'c'd') has been built as shown in figure **7**, gives

$$a^2 + d^2 - 2ad.\cos\varphi = \overline{AB_0}^2 = b^2 + c^2 - 2bc.\cos\mu \tag{5}$$

Whence after substitution apparently

$$e = \tfrac{1}{2}(a^2 + d^2 - b^2 - c^2)/d = \text{constant.} \tag{6}$$

Thus, since the eccentricity remains a constant, point B' has to trace a straight-line running normal to the frame, that is presuming the reflected similar four-bar on top of the initial one, *remains* reflected similar. So, all in all we proved that the motion of the bar B'A' is in fact a *rectilinear translation* in the mentioned direction.

At this point, it is easily seen that $\sphericalangle A_0AB' \equiv \sphericalangle C_0'C'B'$, whence the five-bar $A_0AB'C'C_0'$ alone represents a constrained linkage-pentagon. It is further possible to find a singular bar interconnecting the sides A_0A and $C_0'C'$ of that pentagon. To obtain that bar, we observe the pantograph $B_0BC'C_0'$, simultaneously taking A_0A as the fixed link. Point B of that pantograph then traces a circle about A with respect to A_0A, whence equally the intersection (A_0B x $C_0'C'$) of A_0B and $C_0'C'$ traces a circle about the intersection of A_0A and a line running parallel to AB but meeting the point (A_0B x $C_0'C'$). The resulting six-bar containing the pentagon and the adjoined bar appears to be Hart's 2nd straight-line linkage mechanism since among others B' traces a straight-line.

Multiplying the six-bar with the factor $\overline{A_0D}/\overline{A_0B'}$ in which D = A_0B' x B_0A' then results into an enlarged Hart's 2nd straight-line mechanism, also shown in figure **9**. This mechanism then consists of the pentagonal linkage A_0EDCB_0 and the bar AB. Naturally, turning-joint D of the pentagon similarly traces a straight-line like B' did in the initial six-bar.

The six-bar obtained this way, is to be set up by the initial four-bar A_0ABB_0 and the additional linkage-dyad CDE. (Note that *any* four-bar contains such a unique point D. The point plays a center-rôle in further generalization.)

It is quite possible to turn the 10-bar into an 8-bar linkage. To carry this out, one multiplies the reflected four-bar $A_0'A'B'B_0'$ geometrically from Hart's dyad-joint D. (figure **4**) The multiplication-factor is then decided by the choice of, for instance, B" at A_0D. Whence the factor equals **DB"/DB'**.

Thus, $\square\, A_0''A''B''B_0'' = (\mathbf{DB''/DB'}).\square\, A_0'A'B'B_0'$, (7)

Note that corresponding vertices of these two four-bars remain at the same ray joining the ray-center D.

The initial four-bar (A_0ABB_0) and the obtained one ($A_0''A''B''B_0''$) are further interlinked at the common turning-joints E" and C", being the intersections of corresponding sides. Thus, E" = A_0A x $B''B_0''$ and C" = B_0B x $A_0''A''$. (The connectivity at E" for example is due to the fact that the eccentric slider-crank A_0AB' has been turned into the similar one $A_0E''B''$ through the similarity center A_0 of these sub-chains.) In fact, the former common joint A has now moved to E" while former common joint B moved to C".

Clearly, the bar A"B" moves perpendicular to the frame. Now, the parallel motion of A"B" doesn't have to be sustained by linkage parallelograms, needed earlier for the motion of A'B'. Thus, an eight-bar is obtained with a rectilinear motion of A"B" moving normal to the frame.

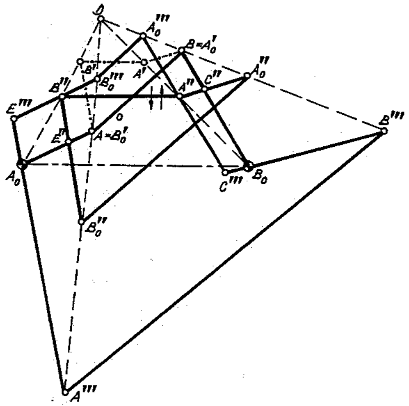
However, in case the rectilinear motion is badly transmitted - as may be the case sometimes - one still has the possibility to adjoin linkage parallelograms to sustain the motion. To avoid the then appearing overconstrainedness, one may simultaneously omit for example the superfluous bar $A_0''B_0''$. The result is a 10-bar linkage to be recognized in figure **4**.

4. The cognate 8-bar

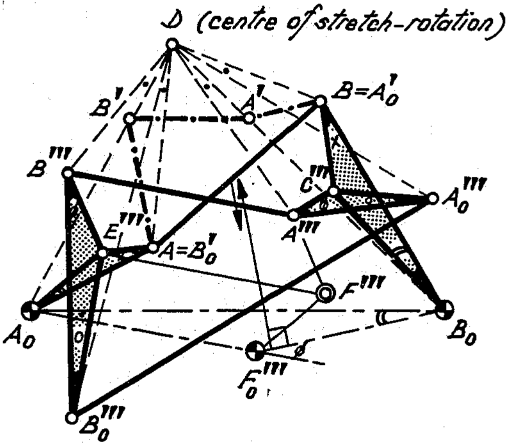
Hart's 2^{nd} straight-line mechanism occurs in pairs. That is to say, for each mechanism of this type, a *curve* cognate exists producing the same part of a straight-line. (figure **2** and also figure 9 of ref.[2].)

Both cognates may further be generalized in a way as demonstrated in the figures **4** and **10**. However, it is quite possible to let them also have the **same** (rectilinear) moving bar A"B". To attain this, one takes for both generalizations the same point B" at the ray A_0D (and/or the same point A" at DB_0.). The two generalizations then appear to be each others' *coupler* cognate producing the identical (rectilinear) motion for the common bar A"B". Figure **11** demonstrates the two of them in one figure, whereas figures **4** and **10** show them separately.

So, for each point B" at A_0D, **two** 8-bar coupler cognates exist, producing the same straight motion perpendicular to the fixed link they have in common.



Two 8-bar coupler cognates having a common bar A''B'' moving perpendicular to the common frame A_0B_0 Figure **11**



random choice for E''' (or C''')

Generalized 8-bar with link A'''B''' moving normal to B_0F_0''' Figure **12**

5. Special Cases

All 6-, 8- and 10-bar straight-line mechanisms of the focal type, have the advantage of a *random* choice for the dimensions of the initial four-bar A_0ABB_0 contained in them. Naturally, particular examples are found by taking specific four-bars to start the design with. If, for instance, the initial four-bar resembles a *deltoid*- linkage, possessing two pairs of adjacent equal sides, the well-known 10-bar linkage of A.B.**Kempe** is obtained. (ref.[7]). Note that in this case AB resembles the angle-bisector of the angle A_0AB'.

In case the initial four-bar A_0ABB_0 represents a contra-parallelogram, the adjoined four-bar

$A_0'A'B'B_0'$ coincides with the initial one. So, the design collapses. Section 2 then shows how to proceed in that case.

6. Generalization through stretch-rotation

In figure **4** point E" joined the input-crank A_0A. It is still possible, however, to generalize the design by a completely random choice of this point. See, for instance figure **12** in which we took the point E''' as a random point attached to the input-crank A_0A. The corresponding design of the generalized 8-bar straight-line mechanism having a rectilinear moving bar would then read something like:

a. Start with the *random* choice of the *4-bar* A_0ABB_0,
b. Make the four-bar $\square$ BA'B'A reflected similar with the initial 4-bar A_0ABB_0,
c. Determine the intersection point $D = A_0B' \times B_0A'$,
d. choose a *random point* E''' in the moving plane of the input-crank A_0A,
e. Stretch-rotate A_0AB' about A_0 into $A_0E'''B'''$ determining point B''',
f. Stretch-rotate $\square$ BA'B'A about D into the 4-bar $A_0'''A'''B'''B_0'''$,
g. Form the similar and rigid triangles B_0BC''' and A_0AE''',
h. Finally, form the similar and rigid triangles $B_0'''B'''E'''$ and $A_0'''A'''C'''$,
i. Note, that $\measuredangle B_0'''E'''B''' = \measuredangle AE'''A_0 = \measuredangle A_0'''C'''A''' = \measuredangle BC'''B_0$,

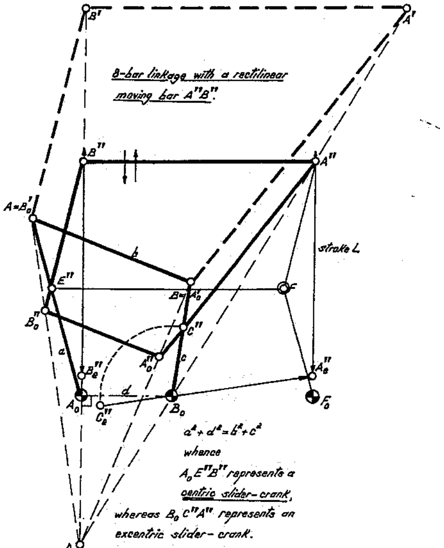
The choice of E''' has the advantage of choosing the motion-direction of the translating bar $A'''B'''$. This bar moves in a fixed direction under an angle of $(\pi/2 + \measuredangle E'''A_0A)$ with A_0B_0. The length of the straight-line is governed by the utmost positions of the eccentric slider-crank $A_0E'''B'''$. Whence, the *length* L of the straight-line, traced by the point B''', is determined by the values of $(\overline{A_0E'''} + \overline{E'''B'''})$, $|\overline{A_0E'''} - \overline{E'''B'''}|$ and by the eccentricity *e'*

In order to sustain the rectilinear motion it is allowed to adjoin the three bars $A'''F'''$, $E'''F'''$ and $F_0'''F'''$ such that $E'''B'''A'''F'''$ as well as $A_0E'''F'''F_0'''$ form linkage-parallelograms. After omitting either one of the bars AB or $A_0'''B_0'''$, one obtains a ten-bar linkage-mechanism still containing a bar always moving perpendicular to a fixed line ($F_0'''B_0$) in the frame, as proved hereafter:

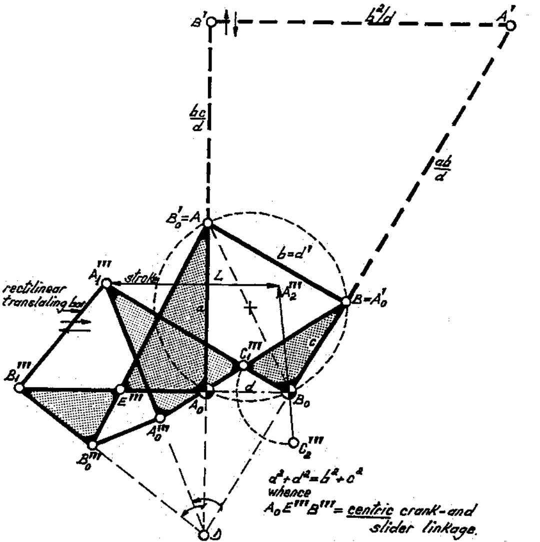
Since for the 5-bar $F_0'''F'''A'''C'''B_0$, the $\measuredangle F_0'''F'''A''' \equiv \measuredangle B_0C'''A'''$, the rectilinear motion of A''' will be perpendicular to $F_0'''B_0$, the frame of the 5-bar. (See Wunderlich's proof with isotropic coordinates in ref.[9])

Further, as the 5-bar $F_0'''F'''A'''C'''B_0$ represents part of the focal mechanism, it is possible to find a bar interlinking the cranks $F_0'''F'''$ and B_0C'''. This bar too may replace the then superfluous bars AB and $A_0'''B_0'''$, again creating another 10-bar linkage with a rectilinear moving bar. The mechanism so obtained then represents Hart's 2nd straight-line mechanism adjoined with two linkage parallelograms, $A_0E'''F'''F_0'''$ and $E'''B'''A'''F'''$, in order to produce the parallel motion for the rectilinear moving bar.

Thus, from the 8-bar, many alternative 10-bar linkages are to be derived producing the same rectilinear motion with sometimes better transmission.
The designer may choose which one he prefers.



$a^2+d^2=b^2+c^2$
whence
$A_0E''B''$ represents a centric slider-crank,
whereas $B_0C''A''$ represents an excentric slider-crank.
Figure 13



$a^2+d^2=b^2+c^2$
whence
$A_0E'''B'''$ = centric crank-and slider linkage.
Example with horizontal motion for bar $A'''B'''$.
Figure 14

7. Zero eccentricity

In the particular case for which B''' traces a straight-line passing through A_0, the eccentricity e' of the slider-crank $A_0E'''B'''$ equals zero, whence e = 0 and so
$a^2+d^2 = b^2 + c^2$, since

$$e' = (\overline{A_0E'''}/\overline{A_0A}).e = \overline{A_0E'''}\ (a^2+d^2-b^2-c^2)/(2ad) \tag{8}$$

(See figure **13** for vertical translation and figure **14** for a horizontal one.)
Then, the initial four-bar has equal transmission angles ($\mu_1 = \mu_2$) in the two aligned positions of the crank A_0A with the frame A_0B_0.
And, since e = 0, we find that

$$\overline{E'''B'''}/\overline{A_0E'''} = \cos\varphi\ /\ \cos\mu = (bc)/(ad) \tag{9}$$

If, *additionally*, $\overline{E'''B'''} = \overline{A_0E'''}$, we find that $\mu \equiv \pm\ \varphi$ and bc = ad (10)
Then, the initial four-bar has to be a **kite**, whereas the bar B'''E''' generates the *elliptic motion.*

8. Historical Note on Linearizers (i.e. *linkages with a rectilinear moving link*)

Apart from the mentioned literature of Hart and Kempe, a booklet of the russian L.D. **Ruzinov** (ref.[10]) has been devoted to this subject. He mentioned more inventors, such as **Gagarin** and others, but all linkages of this kind seem to have been restricted to those containing either deltoids (i.e. kites) or contra-parallelograms.

Ruzinov, for example, wrote, quote: "It is not possible to produce a linearizer based on a four-bar chain", unquote, (page 61 of ref.[10]). Though he searched for a more general approach, he didn't succeed and really thought that such was not possible and only very particular linkages were able to meet the rectilinear motion. That in reality, a *random* four-bar can be used as a base to start the design of a linearizer with, he never uncovered.

Remarkable is also that even Kempe, who founded the generalization of his *compound linkage,* later named the focal linkage by L.Burmester, did not saw the connection between his three kite- or *deltoid-linearizers* and his own interconnected, so-called *conjugate four-piece linkages*. Special cases of the latter were either restricted to Hart's 2nd straight-line mechanism or to other more general linkages, but not to linearizers, having a complete link moving rectilinear.

We conclude that the general approach as developed in the underlying manuscript, really represents a generalized method simultaneously interconnecting all linearizers known to mankind.

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