# Logic, Logics, and Logicism

Solomon Feferman

In memory of George Boolos

**Abstract**

The paper starts with an examination and critique of Tarski's well-known proposed explication of the notion of logical operation in the type structure over a given domain of individuals as one which is invariant with respect to arbitrary permutations of the domain. The class of such operations has been characterized by McGee as exactly those definable in the language $L_{\infty,\infty}$. Also characterized similarly is a natural generalization of Tarski's thesis, due to Sher, in terms of bijections between domains. My main objections are that on the one hand, the Tarski-Sher thesis thus assimilates logic to mathematics, and on the other hand fails to explain the notion of same logical operation across domains of different sizes. A new notion of homomorphism invariant operation over functional type structures (with domains $M_0$ of individuals and $\{T, F\}$ at their base) is introduced to accomplish the latter. The main result is that an operation is definable from the first-order predicate calculus without equality just in case it is definable from homomorphism invariant monadic operations, where definability in both cases is taken in the sense of the $\lambda$-calculus. The paper concludes with a discussion of the significance of the results described for the views of Tarski and Boolos on logicism.

What follows is the text of an invited lecture that I gave at the Boolos Symposium held at the University of Notre Dame April 16–18, 1998. [1] I was very pleased to be a part of that conference in George's memory. I valued him as a friend and as a colleague, for his devotion to logic and philosophy, for his clarity and sense of purpose, and for his unique combination of seriousness and a dry but charming wit. He left an important legacy in many ways that was well demonstrated by the variety of talks at the symposium. It was difficult for me to choose a topic to talk about among our areas of mutual interest, but I finally settled on the one that I think was most fundamental to George's thought, at least since the mid 80s.

I have been ruminating for the last few years about the perennial problem of the precise demarcation of logic—wondering if it is possible to arrive at such a dividing line at all—and that is what is being chewed over here. If you get the feeling that I don't have a settled view of the matter, you are right. But—unlike George—I have been moving more and more to the position that the classical first-order predicate logic has a privileged role in our thought, and so I have been looking at various arguments which justify or challenge that position. Surely some, if not much of this is familiar to the reader, and I apologize for that; but there are new things to say, and they have to be related to what isn't new.[2]

The immediate stimulus for these ruminations was the article by Vann McGee in the 1996 *Journal of Philosophical Logic*, entitled "Logical opera-

[1]It was only at the Notre Dame conference that I learned of the forthcoming publication by Harvard Press of a collection of George Boolos' papers under the title, *Logic, Logic and Logic* (Boolos 1998). I am not sure I would have selected my own title for this lecture if I had known that in advance, but having done so, I am glad that they resonate with each other. (The volume, edited by Richard Jeffrey, with an introduction and afterword by John Burgess, was laid out by Boolos shortly before his death in 1996; it appeared very soon after the conference.)

[2]A first version of the material in this paper was presented at the Sixth CSLI Workshop on Logic, Language and Computation held at the Center for the Study of Language and Information at Stanford University on May 30, 1997, under the title "Logical operations according to Tarski, McGee and me", and again as the First Spinoza Lecture at the 1997 European Summer School in Logic, Language and Information (ESSLLI '97) in Aix-en-Provence, on Aug. 13, 1997 under the title "What is a logical operation?" Only the material of Sections 1–3 agree with the previous presentations. In particular, the proposed notion of similarity invariant operations introduced in Section 4 and the main results there about it and homomorphism invariant operations are new, as is the discussion in Sections 5 and 6. I wish to thank Johan van Benthem for useful early conversations about this work, and Michael Detlefsen and Geoffrey Hellman for useful comments on a draft of this paper.

tions" (McGee 1996). What McGee did there was provide a complete characterization of the proposal—in terms of permutation invariant operations—made by Alfred Tarski in his posthumous publication "What are logical notions?" that appeared ten years earlier (Tarski 1986) (summarized in Section 1 below). Very briefly, if Tarski's proposal is accepted, McGee's work (described in Section 2) shows that the logical operations are exactly those definable in the full infinitary language $L_{\infty,\infty}$.[3] Since this result struck me as blatantly implausible (for reasons given in Section 3), I began to examine the proposal to see why it goes so far afield from what I would have expected, and then I tried to modify it in a way that would come closer. Part of what is presented here is devoted to a new approach in terms of homomorphism invariant operations over functional type structures (Section 4 below) which is somewhat in the spirit of Tarski's basic set-up but gives quite different answers. The main result is that an operation is definable from the first-order predicate calculus without equality just in case it is definable from homomorphism-invariant monadic operations, where definability is taken in the sense of the $\lambda$-calculus. Whether that (or any other invariance notion) can be justified on fundamental conceptual grounds is another matter, certainly in need of pursuit. Section 5 presents certain arguments in its favor, but no principled defense is attempted here. The paper concludes in Section 6 with a discussion of the relation of this work to Tarski and Boolos on logicism.

I do not plan to go into the details of McGee's work, which is faultless in its execution. It is not *that* that is at issue; rather it is *how* to formulate the conceptual problem raised by Tarski which should be the center of our attention. Now, one may well ask why that even matters. After all, we have a plethora of interesting logics suited to a variety of purposes in mathematics, philosophy, linguistics and computer science (and perhaps other fields), and it may seem perverse to try to single out just one of these as the *only* one deserving the unqualified name 'logic'. Well, clearly one difference that would make concerns the logicist program, which stands or falls according to the answer; and, as already noted, I take that up in the conclusion of the paper. I believe another, more important, motivation is to contribute, if possible, to explanations of "how the mind works". Conversely, one may try to single

[3]McGee also used this result to characterize in related terms a natural generalization of Tarski's proposal in terms of isomorphism invariant operations due to Gila Sher (1991); this will figure more prominently below. Section 3 contains a critique of the Tarski-Sher thesis.

out one logic from some answer to *that* question, not on the basis of how men actually reason but how they strive to do so. A quotation from Frege's *Grundgesetze* is apropos: he said that the laws of logic are "the most general laws, which prescribe universally the way in which one *ought* to think if one is to think at all" [itals. mine]. In any case, my ideas in that direction are very preliminary, and I do not try to develop them here at all.

As traditionally defined, "logic is concerned with the principles of valid inference" (Kneale and Kneale 1962, p. 1), or as "the science that investigates the principles of correct or reliable reasoning"(Random House Unabridged), etc., but such definitions leave wide open the exact scope of our subject. Namely, *what are the basic notions with which this science deals*—that is exactly Tarski's question—and *how are the principles of inference featuring these notions to be described?* Now there is immediately a subsidiary methodological question to be raised. Namely, *in which of the following terms is an answer to be sought: model-theoretic (or set-theoretically semantic), proof-theoretic, constructive, or still other*? Not surprisingly, Tarski's approach is entirely within a semantic framework [4]. Moreover, he does not ask, "What is a logical inference?" in the sense of, "What *formal* rules of inference are justified logically?". For—in view of his famous article "On the concept of logical consequence"(Tarski 1936)—*that is the wrong question to ask*, or at least it gives misleading emphasis to the *syntactic* side of logic. Simply, according to his 1936 analysis, " The sentence $\varphi$ *follows logically* from the sentences of the class $K$ if and only if every model of the class $K$ is also a model of the sentence $\varphi$." So, I begin by following Tarski down the path of his half-century later publication.

# 1 Tarski on logical notions.

The circumstances of the posthumous publication (Tarski 1986), edited by John Corcoran, are described in the editor's introduction thereto. Except for minor editorial corrections and amplifications by bibliography and footnotes, it reproduces a typescript for a lecture that Tarski gave—under the same

[4] Tarski's 1986 paper is reprinted in the interesting and useful collection (Shapiro 1996), which consists of essays largely devoted to the issue of the limits of logic from a semantic point of view. For example, one of its main sections concerns the question whether second order logic is to be counted as logic. Several of Boolos' papers on this topic and on plural quantification are also reprinted there, as well of course in (Boolos 1998).

title as that of the paper, "What is a logical notion?"—at Bedford College in London in 1966, and again at SUNY Buffalo in 1973. Tarski asked Corcoran to edit it for publication, but that was only a year before his death in 1983, and he had no further input to its final form.

Tarski's explication in that paper of the concept of logical notion is by an extension to the domain of logic of Klein's *Erlanger Programm* (Klein 1872) for the classification of various geometries according to invariants under suitable groups of transformations. Thus, for example, the notions of metric Euclidean geometry are those invariant under isometric transformations, those of non-metric Euclidean geometry are those invariant under similarity transformations, those of "descriptive" geometry under affine transformations, those of projective geometry under projective transformations, etc., etc. In more modern structural terms, such groups of transformations are simply determined as the automorphisms of structures in a given similarity class **K**, and the notions appropriate to the part of mathematics encapsulated in **K** are then just those invariant under all such automorphisms. Thus, for example, algebra may be considered the study of notions invariant under automorphisms of such structures as rings, fields, and so on, while topology may be considered the study of notions invariant under (auto-)homeomorphisms of topological spaces. Now, Tarski's Kleinean-style explication of logical notions simply takes **K** to be the collection of *all structures of classes and relations of finite type over a basic domain of individuals* $M_0$, and the associated transformations to be just those induced at higher types by arbitrary permutations of $M_0$.[5]

The following spells this out somewhat more precisely (that was not done in Tarski's lecture). First, generate *finite relational type symbols* $\tau$ from the type symbol 0 by successive formation of $\tau = (\tau_1, \dots, \tau_n)$. Then associate with each such $\tau$ a domain $M_\tau$, beginning with an arbitrary class $M_0$ of "individuals" at type 0, by taking $M_\tau$ for $\tau = (\tau_1, \dots, \tau_n)$ to consist of all subrelations $R$ of $M_{\tau_1} \times \cdots \times M_{\tau_n}$. In particular, when $n = 1$, $M_\tau$ is just the class of all subclasses of $M_{\tau_1}$. The structure $M = \langle M_\tau \rangle$ is taken to consist of all the domains $M_\tau$ with the membership relations $\in_\tau$ for each type $\tau$. Now, every permutation $\pi$ of $M_0$ induces in a natural way a permutation of each $M_\tau$ in such a way as to preserve the membership relation at that

[5]Tarski seems to have been unaware of the first proposal of that type for logic by F.I. Mautner (1946). In any case, Mautner pursued the idea in a somewhat different direction from the one taken by Tarski.

type. According to Tarski's explication, by a *logical notion* associated with $M$ is meant a member of one of the $M_\tau$ which is invariant under every such induced permutation.

In his lecture, Tarski gave several simple examples of logical notions in this sense, where by *individual*, is meant a member of $M_0$:

(i) No individual is a logical notion, assuming there are at least two individuals.

(ii) The only classes of individuals which are logical are the empty class and the universal class.

(iii) The only binary relations between individuals which are logical are the empty relation, the universal relation, the identity relation and its complement.

(iv) At the next level, i.e. classes of classes of individuals, Tarski mentioned as logical notions those given by cardinality properties of classes, and says that "the only properties of classes (of individuals) which are logical are properties concerning the number of elements in these classes. That a class consists of three elements, or four elements ... that it is finite, or infinite—these are logical notions, and are essentially the only logical notions on this level."

(v) Finally, among relations between classes (or individuals) Tarski pointed to several which are interesting and "well known to those of you who have studied the elements of logic" such as "inclusion between classes, disjointness of two classes, overlapping of two classes", and so on. He continued: "all these are example of logical relations in the normal sense, and they are also logical in the sense of my suggestion."

Tarski did not attempt to give examples of logical notions in higher types than those in (iv) and (v) in his lecture, nor did he raise the question of characterizing the logical notions there. This is understandable in view of the general audience to which it was addressed. Note that he thus did not go beyond $\tau$ of type level 2 in his illustrations, where

$$lev(0) = 0 \quad \text{and} \quad lev(\tau_1, \ldots, \tau_n) = max(lev(\tau_1), \ldots, lev(\tau_n)) + 1.$$

Before going into McGee's characterization, I want to say something about the significance of Tarski's proposal as it relates to the background in his own work. In his 1936 paper with Lindenbaum, "On the limitations of the means of expression of deductive theories", the first theorem stated is that every relation definable in the simple theory of types is provably invariant under any permutation of the domain of individuals (Lindenbaum and Tarski 1936, p. 385 in Tarski 1983). Next, in his 1936 article "On the concept of logical consequence", Tarski pointed out the following problem concerning his model-theoretic explication of the concept at issue:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage. ... In order to see the importance of this problem for certain general philosophical views it suffices to note that the division of terms into logical and extra-logical also plays an essential part in clarifying the concept 'analytical'. (Tarski 1936, as translated in Tarski 1983, pp. 418-419)

At the time, Tarski concluded his discussion of this problem on a pessimistic note, entertaining the possibility that there would never be definitive results settling the exact boundary between logical and extra-logical notions. It was exactly this problem to which the Bedford College lecture was addressed, thirty years later. We shall return to how this affected his view of what mathematical notions are analytical in the final section below.

The idea of permutation invariance as a criterion for logicality in a semantical setting is a *prima facie* natural one, and has been widely influential, if not explicitly following Tarski's thesis, at least implicitly. Some sample references are: in model theory, (Mostowski 1957), (Lindström 1966), and various of the chapters in (Barwise and Feferman 1985); in linguistics, (Barwise and Cooper 1981), (Keenan and Stavi 1986), (van Benthem 1989), (Keenan 1999); and in the philosophy of logic, (Peacocke 1976), (McCarthy 1981), (Simons

1988), (Sher 1991), and various of the papers in (Shapiro 1996). In particular, Tarski's thesis is embraced wholeheartedly (with a natural modification to invariance across bijections between domains, as will be discussed below) in Gila Sher's book just referred to.

## 2 McGee's characterization of logical operations.

What is characterized in (McGee 1996) are those operations $O(\langle A_i \rangle_{i<\gamma}) = B$ on sequences of relations $A_i$ between individuals to relations $B$, which are invariant under arbitrary permutations of the universe $M_0$ of individuals. McGee allows the number of argument places $\gamma$ of $O$ to be an infinite ordinal, and each $A_i$ is allowed to have an infinite number of argument places. To simplify matters technically, he takes the relations in question to be subsets of the set $M_0^V$ of all variable assignments $\sigma : V \to M_0$, where $V$ is a non-empty set of variables. For $n > 0$ let $(n)$ be the type of $n$-ary relations on $M_0$, so that $\tau = ((n_1), \ldots, (n_k))$ is the type of $k$-ary relations $P$ between relations $R_1, \ldots, R_k$, where $R_i$ is $n_i$-ary. Tarski's notion of logical relation of type $\tau$ may be reduced to McGee's notion of logical operation, as follows. Given a set $A$ of variable assignments $\sigma$, and a natural number $n$ with $0 < n \leq card(V)$, let $A|n$ be the set of $(a_0, \ldots, a_{n-1})$ such that there is a $\sigma : V \to M_0$ with $\sigma(x_i) = a_i$, for each $i < n$ and $\sigma \in A$, where $x_0, \ldots, x_{n-1}$ are the first $n$ variables of $V$. Now assume each $n_i \leq card(V)$. Then take $O_P$ to be the operation given by

$$\sigma \in O_P(A_1, \ldots, A_k) \text{ iff the relation } P \text{ holds between } (A_1|n_1, \ldots, A_k|n_k).$$

In other words, the value of $O_P$ at a sequence of sets of $V$-assignments is the universal set of $V$-assignments if $P$ is true of the associated relations, and is the empty set otherwise. Then $P$ is invariant under arbitrary permutations of the basic domain (in Tarski's sense) just in case $O_P$ is invariant under arbitrary permutations of it (in McGee's sense). In his main characterization theorem, McGee considers only operations $O$ of type level 2, but later describes how to extend his result to operations of higher type level.

The formulas of the language $L_{\infty,\infty}$ are generated from given atomic formulas by the operations of negation, arbitrarily long conjunctions and disjunctions, and arbitrarily long universal and existential quantifier sequences (cf., e.g., Karp 1964). An operation $O$ on a sequence $A = \langle A_i \rangle_{i<\gamma}$ of subsets of $M_0^V$ is said to be *definable in* $L_{\infty,\infty}$ , if there is a formula $\varphi$ of that language with basic relation symbols $=$ and $V$-ary $\mathbf{A}_i$ for each $(i < \gamma)$ such that

$$(M_0, A) \models \varphi[\sigma] \quad \text{iff} \quad \sigma \in O(A),$$

where each symbol $\mathbf{A}_i$ is interpreted by the set $A_i$ for each $(i < \gamma)$ in the structure $(M_0, A)$.

**Theorem 1** *(McGee 1996, p. 572). $O$ is invariant under arbitrary permutations of the domain of individuals if and only if $O$ is definable in $L_{\infty,\infty}$.*

It is straightforward that every $L_{\infty,\infty}$ definable operation is invariant under arbitrary permutations of the domain of individuals. The idea of the proof in the other direction is to lay out all possibilities for the operation $O$ as its arguments range over all possible $A$. This can be achieved using a set $W$ of variables containing $V$, with $card(W - V) = \kappa + 1$. Enumerate $M_0$ as $\{s_\alpha : \alpha < \kappa\}$ and $W - V$ as $\{x_a : \alpha < \kappa\} \cup \{y\}$. The $x_\alpha$ act as formal surrogates of the $s_\alpha$. Let $\psi_A$ be the diagram of $A$ under this association together with $\neg(x_\alpha = x_\beta)$ for each $\alpha < \beta$, and then take $\chi_A$ to be the formula which says that whenever $x_\alpha$ $(\alpha < \kappa)$ are such that $\psi_A$ holds and each $y$ in the domain is one of the $x_\alpha$, then (using $V$) the formal expression of $\sigma \in O(A)$ holds. Finally, take $\varphi$ to be the conjunction of all the $\chi_A$ over all sequences $A$ of the considered kind; note that this final conjunction is of cardinality at least $2^\kappa$, and the longest quantifier sequences in $\varphi$ are of length at least $\kappa$.

McGee says (p. 575) that this theorem "gives us good reason to believe that the logical operations on a particular domain are the operations invariant under permutations". It is natural, though, not to tie logical operations to specific domains, and so he goes on to consider operations *across domains* which for each set of individuals $M_0$ turn sequences $A$ of subsets $A_i$ of $V$-assignments in $M_0$ into a subset $B$ of $V$-assignments in the given domain. Then he argues (rightly, in my view), that "(i)n order for an operation across domains to count as logical, it is not enough that its restriction to each particular domain be a logical operation". For example, McGee defines an operation of "wombat disjunction" $\cup_W$ across domains such that $\sigma$ belongs to $A_1 \cup_W A_2$ iff there are wombats in the universe of discourse $M_0$ and $\sigma \in A_1 \cup A_2$ or there are no wombats in the universe of discourse and $\sigma \in A_1 \cap A_2$. Clearly wombat disjunction is not a logical operation, though on each domain it is invariant under permutations. Another example given is that of "affluent cylindrification" $\$(A)$ which holds in a domain just in case some rich person belongs to $A$; again this is not a logical operation, but meets the permutation invariance condition on "upper-crust domains" in which every person is rich. However, on an equinumerous domain containing at least one rich and one poor person, the operation \$ is not permutation invariant by taking $A$ to be a singleton of one of these.

Thus McGee is led to consider an extension of Tarski's thesis proposed by Gila Sher (1991, pp. 53 ff) according to which:

> an operation across domains is logical if it is invariant under every bijection between domains.

He refers to this as the *Tarski-Sher thesis*, and I shall follow him in that, with one exception. As already noted, Tarski is apparently willing to consider logical operations at any level of the type hierarchy, while Sher explicitly restricts herself to operations of type level $\leq 2$ (Sher 1991, p. 54, Condition A), so it is only for the latter that their names are legitimately paired. The following is then a corollary of the result above.

**Theorem 2** *(McGee 1996, p. 576). An operation $O$ across domains is a logical operation according to the Tarski-Sher thesis iff for each cardinal $\kappa \neq 0$ there is a formula $\varphi_\kappa$ of $L_{\infty,\infty}$ which describes the action of $O$ on domains of cardinality $\kappa$.*

More specifically, one can take $\varphi_\kappa$ to be the formula constructed for the proof of Theorem 1 for any domain $M_0$ of cardinality $\kappa$. Whatever such $\varphi_\kappa$ is taken, in order to obtain a single definition of the operation $O$ across domains, one must take something like the disjunction over the class of all non-zero cardinals $\kappa$ of the statement that there are exactly $\kappa$ elements in the domain and that $\varphi_\kappa$ holds. This goes beyond $L_{\infty,\infty}$ as ordinarily conceived.

# 3 Critique of the Tarski-Sher thesis

McGee's results lay bare the character of logical operations according to the Tarski-Sher thesis. I have, accordingly, three basic criticisms of it:

1) The thesis assimilates logic to mathematics, more specifically to set theory.

2) The set-theoretical notions involved in explaining the semantics of $L_{\infty,\infty}$ are not robust.

3) No natural explanation is given by it of what constitutes the *same* logical operation over arbitrary basic domains.

The first of these, I think, speaks for itself, given McGee's results, but it will evidently depend on one's gut feelings about the nature of logic as to whether this is considered reasonable or not. For Sher, to take one example, this is no problem. Indeed, she avers that "(t)he bounds of logic, on my view, are the bounds of mathematical reasoning. Any higher-order mathematical predicate or relation can function as a logical term, provided it is introduced in the right way into the syntactic-semantic apparatus of first-order logic." (Sher 1991, pp. xii-xiii) What that "right way" is for her, is spelled out in a series of syntactic/semantic conditions A–E (op. cit pp. 54-55), of which the crucial ones are condition A that a logical operation is of type-level at most 2, and E is the condition for invariance under bijections. The paradigms of condition A are the cardinality quantifiers of (Mostowski 1957) and, more generally, the generalized quantifiers of (Lindström 1966), where the bound variables range over individuals of the domain. But note that $L_{\infty,\infty}$ accommodates second-order quantification as a logical operation across domains (in the Tarski-Sher sense) as follows. First, given formulas $\psi(X)$ and $\theta(x)$ of this language, where $X$ is a second-order variable, by $\psi(\{x : \theta(x)\})$ is meant the result of substituting $\theta(t)$ for each occurence of an atomic formula $t \in X$ in $\psi$. Thus, on a domain of cardinality $\kappa$, $(\forall X)\psi(X)$ is equivalent to the statement $\varphi_\kappa$ that there exist $\kappa$ elements $x_\alpha$ which are distinct and exhaustive of the domain, and are such that

$$\bigwedge_{S\subseteq\kappa} \psi(\{y : \bigvee_{\alpha\in S} y = x_\alpha\}) \text{ holds.}$$

(Again, we require a conjunction of cardinality $2^\kappa$ in this formula.) As a logical operation $O$ of type-level 2 across arbitrary domains $(\forall X)\psi(X)$ is defined by the disjunction of all these $\varphi_\kappa$ over all non-zero cardinals $\kappa$. So, from Theorem 2 above, the restriction to bound first order variables is only apparent, and Sher's condition A is not set-theoretically restrictive. By a trick similar to the preceding, we can quantify over arbitrary relations on the domain, and then say that they are functions, etc. In particular, we can express the Continuum Hypothesis and many other substantial mathematical propositions as logically determinate statements on the Tarski-Sher thesis. Of course, if one follows Tarski by allowing consideration of invariant notions in all finite types, the assimilation of logic to set theory is patent on his thesis, without needing to invoke infinite formulas at all. But insofar as one or the other version of the thesis requires the existence of set-theoretical entities of a special kind, or at least of their determinate properties, it is evident that we have thereby transcended logic as the arena of universal notions independent of "what there is".

The point 2) is in a way subsidiary to 1). The notion of "robustness" for set-theoretical concepts is vague, but the idea is that if logical notions are at all to be explicated set-theoretically, they should have the same meaning independent of the exact extent of the set-theoretical universe. For example, they should give equivalent results in the constructible sets and in forcing-generic extensions. Gödel's well known concept of absoluteness provides a necessary criterion for such notions, and when applied to operations defined in $L_{\infty,\infty}$, considerably restricts those that meet this test. For example, the quantifier "there exist uncountably many $x$" would not be logical according to this restriction, since the property of being countable is not absolute. The study of absolute logics was initiated by (Barwise 1972), and is surveyed in detail in (Väänänen 1985). One should be aware that the notion of absoluteness is itself relative, and is sensitive to a background set theory, hence again to the question of what entities exist. For examples of absolute operations which are patently set-theoretical yet come out as logical on the Tarski-Sher thesis, see the just mentioned reference.

For me, point 3) is perhaps the strongest reason for rejecting the Tarski-Sher thesis, at least as it stands. It seems to me there is a sense in which the usual operations of the first-order predicate calculus have the *same meaning* independent of the domain of individuals over which they are applied. This characteristic is *not* captured by invariance under bijections. As McGee puts it "(t)he Tarski-Sher thesis does not require that there be any connections among the ways a logical operation acts on domains of different sizes. Thus, it would permit a logical connective which acts like disjunction when the size of the domain is an even successor cardinal, like conjunction when the size of the domain is an odd successor cardinal, and like a biconditional at limits."(McGee 1996, p. 577) In the end (though perhaps more for other reasons), McGee accepts the Tarski-Sher thesis as a necessary condition for an operation to count as logical, but not a sufficient one. I agree completely, and believe that if there is to be an explication of the notion of a logical operation in semantical terms, it has to be one which shows how the way an operation behaves when applied over one domain $M_0$ connects naturally with how it behaves over any other domain $M'_0$. The concept of homomorphism invariant operation studied in the next section is proposed as a first step in that direction.

# 4 Homomorphism and similarity invariant operations.

It is useful here to expand the relational finite type framework of Sec. 1 to a functional finite type framework, where we add a basic type symbol $b$ for booleans (i.e., truth values).[6] The *functional finite type symbols* (t.s.) are generated from 0 and $b$ by formation of $\tau = (\tau_1, \ldots, \tau_n \to \sigma)$ whenever $\tau_1, \ldots, \tau_n$ and $\sigma$ are t.s. By a *functional type structure* $M = \langle M_\tau \rangle$ over $M_0$ is meant one where $M_b = \{T, F\}$ (or $\{1, 0\}$), and for each $\tau$ as above, $M_\tau$ consists of functions from $M_{\tau_1} \times \cdots \times M_{\tau_n}$ to $M_\sigma$. $M$ is said to be *maximal* if for each such $\tau$, $M_\tau$ is the set of *all* such functions. For simplicity we shall assume that $M$ is maximal in the following, though for most purposes weaker assumptions (including closure under definability in the typed $\lambda$-calculus) suffice. The relational type symbols $(\tau_1, \ldots, \tau_n)$ are identified hereditarily with the *propositional function type symbols* $(\tau_1, \ldots, \tau_n \to b)$, and sub-relations of $M_{\tau_1} \times \cdots \times M_{\tau_n}$ are identified with their characteristic functions. In the following we shall write $\overline{\tau}$ for $(\tau_1, \ldots, \tau_n)$, $M_{\overline{\tau}}$ for $M_{\tau_1} \times \cdots \times M_{\tau_n}$ and $\overline{x}$ for $(x_1, \ldots, x_n)$ in $M_{\overline{\tau}}$.

**Definition 1** *Suppose $M = \langle M_\tau \rangle$ and $M' = \langle M'_\tau \rangle$ are two functional type structures over $M_0$ and $M'_0$ respectively. By a* similarity relation $\sim$ *between $M$ and $M'$ we mean a collection of relations $\sim_\tau$ for each t.s. $\tau$ such that*

(i) $\forall x \in M_0\, \exists x' \in M'_0\, (x \sim x')\ \&\ \forall x' \in M'_0\, \exists x \in M_0\, (x \sim x')$,

(ii) $\forall x \in M_b\, \forall x' \in M'_b\, [\, x \sim x' \Leftrightarrow\ x = x'\, ]$, *and*

[6] Other standard notations for base types are Church's $\iota$ for the type of individuals and 0 for that of truth values, while in categorial grammar of natural language it is frequent to use $e$ for individuals and $t$ for truth values. On the other hand, it is standard in pure type theory to take 0 for the type of individuals, and that dictated my choice of $b$ for booleans.

(iii) *for each* $\tau = (\overline{\tau} \to \sigma)$ *and* $p \in M_\tau$ *and* $p' \in M'_\tau$ *wehave*

$$p \sim p' \Leftrightarrow \forall \overline{x} \in M_{\overline{\tau}} \forall \overline{x'} \in M'_{\overline{\tau}}[\ \overline{x} \sim \overline{x'} \Rightarrow p(\overline{x}) \sim p'(\overline{x'})\ ].$$

(Note that we have dropped the type subscripts to the higher type relations $\sim_\tau$, since these are determined by the context.) Relations satisfying (iii) only are called *logical relations* [sic!] in the typed $\lambda$-calculus literature; cf., e.g. (Mitchell 1990, pp. 416 ff). As with the latter, each similarity relation is completely determined by $\sim_0$ using (ii) and (iii).

When one has a map $h : M_0 \to M'_0$ which is *onto*, the relation

(iv) $$x \sim_0 x' \text{ iff } h(x) = x'$$

determines a similarity relation between $M$ and $M'$. We shall call $h$ a *homomorphism from* $M$ onto $M'$,[7] though $h$ itself is only partially extendible as a mapping to higher types. For example, if $\tau = (0^n \to b)$ and $p \in M_\tau$ and $p' \in M'_\tau$, then

(v) $$p \sim p' \text{ iff } \forall \overline{x}, \overline{x'}(h(\overline{x}) = \overline{x'} \Rightarrow p(\overline{x}) = p'(\overline{x'})).$$

Then for each $p$ there is at most one $p'$ with $p \sim p'$, and we can write $h(p) = p'$ when this is defined, namely if we have $p(\overline{x}) = p(\overline{y})$ whenever $h(\overline{x}) = h(\overline{y})$. Trivially, there is for each $p'$ a $p$ with $h(p) = p'$ in this sense. When $p \sim p'$, we can think of $p'$ as being obtained from $p$ by *shrinking along* $h$.

**Definition 2** *An operation $O$ is* of type $\tau$ across domains *if for each functional type structure $M$ we have an associated $O^M \in M_\tau$. $O$ is said to be* similarity invariant *if for each $M, M'$ and similarity relation $\sim$ between $M$ and $M'$ we have $O^M \sim O^{M'}$. It is said to be* homomorphism invariant*, if this is only required to hold for similarity relations determined by homomorphisms from $M$ to $M'$.*

It seems to me that there is a natural sense in which *operations $O$ invariant under homomorphisms are logical form preserving, if one ignores*

[7] A more appropriate use of mathematical terminology for this notion would be *epimorphism*. As only *onto* homomorphisms are considered here throughout, I did not feel it was necessary to use that less suggestive terminology.

*equality*, at least for propositional operations $O$ of type level 2 with proposition function arguments. This means that whenever $h$ is a homomorphism from $M$ onto $M'$ and arguments $p_i$ are shrunk to $p'_i$ along $h$, then $O^M(p_1, \ldots, p_n) = O^{M'}(p'_1, \ldots, p'_n)$.

To get an idea how much closer homomorphism invariance brings us to definability in first-order logic, consider the following three negative examples.

**1) Equality of individuals.** This is given by the operator $I$ of type $(0^2 \to b)$ defined by

$$I^M(x, y) = [\ T \text{ if } \quad x = y, \text{ else } F\ ]$$

Clearly $I$ is *not* homomorphism invariant.

**2) Cardinality quantifiers.** For each non-zero cardinal number $\kappa$, the quantifier $E_\kappa$ of type $((0 \to b) \to b)$ is defined by

$$E^M_\kappa(p) = [\ T \text{ if there are at least } \kappa \text{ distinct } x \text{ such that } p(x) = T, \text{ else } F\ ].$$

*None* of the operators $E_\kappa$ is homomorphism invariant for $\kappa \geq 2$.

**3) Internal function quantifiers.** The quantifier ${}^2A$ for universal second order function quantification of type $(((0 \to b) \to b) \to b)$ is defined for $f$ of type $((0 \to b) \to b)$ by

$${}^2A^M(f) = [\ T \text{ if } (\forall q \in M_{0\to b}) f(q) = T, \text{ else } F\ ].$$

If ${}^2A$ were homomorphism invariant so also would be the following operation $O$ defined from it in combination with the operations of the first-order predicate calculus, which (as we shall show in a moment) *are* homomorphism invariant. $O$ is of type $((0 \to b) \to b)$, and $O^M(p) = T$ if the extension $\hat{p}$ of $p$, $\{x \mid p(x) = T\ \}$, is the intersection of all the proper extensions $q$ of $p$, i.e. of all those $q$ with $\hat{p} \subseteq \hat{q}$ for which $\exists y(q(y) = T$ and $p(y) = F)$; otherwise $O^M(p) = F$. Let $M_0$ contain three distinct elements $x$, $y$ and $z$, and let $q(x) = T$, $q(y) = q(z) = F$. Let $M'_0 = \{x, y\}$ and $q'(x) = T$, $q'(y) = F$. Then the homomorphism $h(x) = x$, $h(y) = h(z) = y$ sends $q$ to $q'$, but $O^M(q) = T$ while $O^{M'}(q') = F$. Thus ${}^2A$ is *not* homomorphism invariant.

Now we turn, by contrast, to the familiar operations of the first-order predicate calculus (without equality), denoted PC below.

**4) Negation.** The operation $N$ of negation is of type $(b \to b)$ and is defined in each $M$ by

$$N^M(p) = [\ F \text{ if } p = T,\ \text{else } F\ ].$$

$N$ is clearly similarity invariant.

**5) Conjunction.** The operation $C$ of conjunction is of type $(b, b \to b)$ and is defined in each $M$ by

$$C^M(p, q) = [T \text{ if } p = T \text{ and } q = T,\ \text{else } F].$$

$C$ is clearly similarity invariant.

**6) Existential quantification over individuals.** The operation $E$ of existential quantification over the domain of individuals is of type $((0 \to b) \to b)$ and is defined in each $M$ by

$$E^M(p) = [\ T \text{ if } \exists x \in M_0(p(x) = T),\ \text{else } F\ ].$$

$E$ is similarity invariant by the following argument. Suppose $\sim$ is a similarity relation between $M$ and $M'$, and suppose $p, p'$ are of type $(0 \to b)$ in $M$ and $M'$, resp., with $p \sim p'$. Thus, whenever $x \in M_0$ and $x' \in M'_0$ are such that $x \sim x'$ then $p(x) = p'(x')$. If $E^M(p) = T$ then also $E^{M'}(p') = T$; for, given any $x \in M_0$ such that $p(x) = T$, by the first similarity condition there exists $x' \in M'_0$ such that $x \sim x'$, so $p'(x') = p(x) = T$. By symmetry, if $E^{M'}(p') = T$ then $E^M(p) = T$. Hence $E^M(p) = E^{M'}(p')$ for all such $p, p'$, and so $E^M = E^{M'}$.

**Definition 3** *An operation $O$ is said to be* definable from operations $O_1, \ldots,$ *$O_k$ if it is given by a definition from them in the $\lambda$-calculus uniformly over each $M$, i.e. if there is a term $t(z_1, \ldots, z_k)$ of the typed $\lambda$-calculus with constants $T$ and $F$, where each $z_i$ is of the same type as $O_i$ and $t$ is of the same type as $O$, such that in each functional type structure $M$, $O^M = t(O_1^M, \ldots, O_k^M)$.*[8]

[8]Alternatively, we can take $t$ to be a term in the typed combinatory calculus generated by application from the variables $z_1, \ldots, z_k$, the constants $T$ and $F$, and the typed combinators $K$ and $S$ in each appropriate combination of types.

It is easily seen that every operation definable in this way from similarity (homomorphism) invariant operations is again similarity (homomorphism) invariant. The main result, Theorem 6 below, characterizes the operations $O$ definable from the operations of the first-order predicate calculus PC without equality in terms of those definable from homomorphism invariant operations of certain monadic type.

We first have from examples 4)–6) that:

**Theorem 3** *If $O$ is definable from $N, C$ and $E$ then $O$ is similarity invariant.*

**Theorem 4** *Every operation $O$ determined by a formula of PC, the first-order predicate calculus without equality, is definable from $N, C$ and $E$, and hence is similarity invariant.*

**Proof.** This is illustrated by the operation $O$ of type $((0^2 \to b), (0^3 \to b), 0 \to b)$ determined by the formula $\forall x[P(x,z) \to \exists y Q(x,y,z)]$, which we rewrite as

$$\neg\exists x[P(x,z) \wedge \neg\exists y Q(x,y,z)].$$

Then in each $M$ with $p$ of type $(0^2 \to b)$, $q$ of type $(0^3 \to b)$ and $z$ of type 0, we have

$$O^M(p,q,z) = N(E(\lambda x.C(p(x,z), N(E(\lambda y.q(x,y,z)))).$$

The idea is clear: each time one quantifies existentially over a subformula with respect to a variable $x$, in the corresponding term one first $\lambda$-abstracts with respect to $x$, and then applies the operator $E$. The propositional connectives are defined directly in terms of $N$ and $C$. For a general proof one can also make use of (Quine 1971), which basically provides a variable-free form of PC, $\lambda$-definable from $N, C$ and $E$; cf., also, (van Benthem 1991, p. 276). Note that only abstraction with respect to individual variables is here required.

We cannot expect a simple converse to Theorem 4, as the following counter-example shows:

**7) Well-foundedness quantifier.** The test W for well-foundedness of binary relations between individuals is of type $(0^2 \to b) \to b)$. Using $N$ for the set of natural numbers, W is defined by

$$W^M(r) = [\ T \text{ if } (\forall f : N \to M_0)(\exists n \in N)\ r(f(n+1), f(n)) = F, \text{ else } F\ ];$$

i.e. $W^M(r) = T$ just in case there are no infinite descending r-sequences. Then W is similarity invariant. To show this, consider any similarity relation $\sim$ between $M$ and $M'$ such that $r \sim r'$ for $r, r'$ of type $(0^2 \to b)$ in $M, M'$, resp. Given $f : N \to M_0, f' : N \to M'_0$ define $f \sim f'$ if $\forall n(f(n) \sim f'(n'))$: then by the condition $(i)$ on similarity relations, $\forall f \exists f'(f \sim f')$ and $\forall f' \exists f(f \sim f')$. Also, if $f \sim f'$ then $r(f(n+1), f(n)) = r'(f'(n+1), f'(n))$. It then follows easily that $W^M(r) = W^{M'}(r')$, as required.[9]

As we see next, it is essential in the preceeding counter-example that the argument of W is of the type of relations, or binary propositional functions.

**Definition 4** *Take $\pi$ for the type $(0 \to b)$ of unary propositional functions, which correspond to monadic predicates. A type $\tau = (\overline{\tau} \to \sigma)$ is said to be* monadic *if $\sigma = b$ (i.e. it is a propositional function) and each argument type $\tau_i$ is either $\pi$, $b$ or 0. $\tau$ is said to be* pure monadic *if it has the form $(\pi^n \to b)$.*

Note that the operations $N, C$ and $E$ are of monadic type, with $E$ of type $(\pi \to b)$.

**Theorem 5** *Suppose $O$ is of monadic type and is homomorphism invariant. Then $O$ is definable in terms of $N, C$ and $E$.*

**Proof**. We give the proof for the case that the type of $O$ is pure monadic; the general case is obtained by a slight modification to be explained below. In the following we shall deal with structures $(M, \overline{p})$ where $\overline{p} = (p_1, \ldots, p_n)$. In each such, the unary propositional functions $p_i$ determine a partition of $M_0$ whose components are given by a choice of $T$ or $F$ for each $i = 1, \ldots, n$. It is convenient here to identify $T$ with 1 and $F$ with 0; then each component is identified with an $n$-termed sequence $\overline{k}$ of zeros and ones, and an individual $y$ belongs to that component just in case $\overline{p}(y) = \overline{k}$. Each non-empty component can be shrunk to a single point, and this determines a homomorphism of $M$ onto $M'$ sending $\overline{p}$ onto $\overline{p'}$, and is hence such that $O^M(\overline{p}) = O^{M'}(\overline{p'})$. The behavior of $O$ is then determined by its behavior on the finitely many reduced such structures $(M', \overline{p'})$.

[9]The example of $W$ as an homomorphism invariant operation not definable in PC was suggested to me by Johan van Benthem. The contrast with the operation ${}^2A$ of example 3) is interesting; the latter makes use of function quantification *internal* to $M$, while W uses function quantification *external* to $M$.

In more detail, the proof goes as follows. Given $(M, \overline{p})$, let

$$y \equiv z \Longleftrightarrow \overline{p}(y) = \overline{p}(z)$$

and let $h(y) = [y]$, where $[y]$ is the equivalence class of $y$ under the equivalence relation $\equiv$. Let, further,

$$D(M, \overline{p}) = \{ \ \overline{k} \mid (\exists y)\overline{p}(y) = \overline{k} \ \}, \text{ and } [[\overline{k}]] = \{ \ y \mid \overline{p}(y) = \overline{k} \ \}.$$

Then $[[\overline{k}]] \neq \emptyset$ iff $\overline{k} \in D(M, \overline{p})$. Furthermore, if $[[\overline{k}]] \neq \emptyset$ then $y \in [[\overline{k}]]$ iff $[[\overline{k}]] = [y]$, so the non-empty equivalence classes are the same as the $[[\overline{k}]]$ in $D(M, \overline{p})$. Moreover, if $\overline{k} \neq \overline{l}$ then $[[\overline{k}]] \cap [[\overline{l}]] = \emptyset$. So, the image $M_0'$ of $M_0$ under $h$ is in one-one correspondence with $D(M, \overline{p})$. Let $\overline{p}'([y]) = \overline{p}(y)$ for each $y$ in $M_0$; then by the invariance of $O$ under homomorphisms, we have $O^M(\overline{p}) = O^{M'}(\overline{p'})$. Now, consider any other $(M^*, \overline{p^*})$ with $D(M, \overline{p}) = D(M^*, \overline{p^*})$. Defining the equivalence relation and homomorphism on $(M^*, \overline{p^*})$ just as we did above on $(M, \overline{p})$, we see that its homomorphic image is isomorphic to $(M', \overline{p'})$. Hence $O^M(\overline{p}) = O^{M^*}(\overline{p^*})$. This shows that the behavior of $O$ is completely determined by the sets $O^+$ and $O^-$ defined as follows:

$$O^+ = \{ \ D(M, \overline{p}) \mid O^M(\overline{p}) = T \ \} \text{ and } O^- = \{ \ D(M, \overline{p}) \mid O^M(\overline{p}) = F \ \}.$$

Each of these is a finite set of subsets of the set of all $n$-termed sequences of zeros and ones, say

$$O^+ = \{D_1, \ldots, D_r\} \text{ and } O^- = \{D_{r+1}, \ldots, D_s\}.$$

Then $O$ is defined in general by

$$O^M(\overline{p}) = \begin{cases} T & \text{if } \ D(M, \overline{p}) = D_i \ \text{ for some } i = 1, \ldots, r, \text{ and} \\ F & \text{if } \ D(M, \overline{p}) = D_i \ \text{ for some } i = r+1, \ldots, s \end{cases}$$

Finally, given any subset $D$ of the set of all $n$-ary sequences $\overline{k}$ of zeros and ones, the condition $D(M, \overline{p}) = D$ is equivalent to the conjunction of all formulas $\exists y(\overline{p}(y) = \overline{k})$ for $\overline{k}$ in $D$ and of all formulas $\neg\exists y(\overline{p}(y) = \overline{k})$ for $\overline{k}$ in the complement of $D$. Thus $O$ is definable in terms of the operations $N, C$ and $E$; logically speaking, it is given by a formula of the monadic predicate calculus with $n$ monadic predicate symbols.

Now suppose that $O$ is an homomorphism invariant operation of type $(\pi^n, 0^m \to b)$. Here we have to show how $O$ behaves on structures $(M, \overline{p}, \overline{x})$

where $\overline{x} = (x_1, \ldots, x_m)$. Again, one forms "diagrams" $D(M, \overline{p}, \overline{x})$. This includes the information for each $x_j$ that it is in a unique member of the partition determined by $\overline{p}$; moreover, if $x_j \in [[\overline{k}]]$ then the condition $\exists y(\ \overline{p}(y) = \overline{k})$ is superseded by $\overline{p}(x_j) = \overline{k}$. The proof then proceeds as before, and one ends up once more with $O$ definable from $N, C$ and $E$, corresponding logically to a formula in the monadic predicate calculus with $n$ monadic predicate symbols and $m$ free variables. The most general case for an operation of monadic type is that it also contains arguments of type $b$; this simply adds a certain number of purely propositional arguments, whose truth value is invariant under homomorphisms, and thus do not disturb the proof.

**Theorem 6** *The operations definable from the operations of the predicate calculus PC without equality are exactly those definable from homomorphism invariant operations of monadic type.*

**Proof.** By Theorem 4, every operation definable from a PC operation is definable from $N, C$ and $E$, each of which is of monadic type and is similarity invariant, hence is homomorphism invariant. The converse is a corollary of Theorem 5.

# 5 Homomorphism invariance as a criterion for logicality, pro and con.

The discussion in the following is divided up according to a series of issues, particularly concerning identity, cardinality quantifiers, higher-order quantification, and the evidence from natural language. It concludes with a brief pointer to other approaches.

## 5.1 Is identity a logical notion?

If one is sympathetic to seeking a model-theoretic characterization of the operations of first-order logic in terms of some invariance condition, there might still be disappointment in—if not opposition to—the use of homomorphism invariance for that purpose, since it excludes the relation $I$ of identity. Quine, for example, worries over the general issue as to whether identity is to be considered a logical notion in his *Philosophy of Logic* (Quine 1986, pp. 61 ff). On the one hand, he says that it "seems fitting" that the predicate of

$=$ is to be counted with predicates such as $<$ and $\in$ as part of mathematics and not of logic. On the other hand, he gives three arguments for counting $=$ as part of logic. The first is the completeness of the logic of PC with equality, the second is the "universality" of $=$, and the third is the possibility of "simulating" $=$ in a language $L$ containing finitely many predicate symbols; by that he means its explicit definition from those predicates to satisfy the condition of identity of indiscernibles. The first and third arguments are irrelevant to a logicality-cum-invariance project, both because of their character and because they do not treat the operation $I$ independently of the logical operations of the first-order predicate calculus without equality. The second argument is perhaps the most convincing in general terms but does not rely on invariance in the Tarski-Sher sense per se. In the end, it is the third argument that persuades Quine the most. "The upshot is ... that identity theory has stronger affinities with its neighbors in logic than with its neighbors in mathematics. It belongs in logic. Yet we saw it as a threat to our structurally conceived definitions of logical truth ... A reconciliation is afforded [by the definition of $=$ in terms of the other predicates, and then] all laws of identity become mere abbreviations of logical truths of the purely quantificational sort... The structural view of logic is sustained." (op. cit., p. 64). However, this is not an argument in favor of identity as a logical notion in its own right, but rather as a notion that can be reduced in certain contexts to logical notions.

It is undeniable that the relation of identity has a "universal", accepted and stable logic (at least in the presence of totally defined predicates and functions, as is usual in the PC with $=$), and that argues for giving it a distinguished role in logic even if it should not turn out to be logical on its own under some cross-domain invariance criterion, such as under homomorphisms. Of course, even if a form of the latter is accepted as a criterion for logicality, one is still free to consider the operations which are defined *from* $I$ by those provided in Theorem 6. That of course buys one the quantifiers $E_\kappa$ for $\kappa$ finite, but not those for $\kappa$ infinite, whose loss is discussed separately, next.

### 5.2 What about the cardinality quantifiers and second-order quantification?

It seems to me clear that the cardinality quantifiers $E_\kappa$ for $\kappa$ uncountable belong to mathematics (specifically, set theory) and not to logic; they are all excluded by the homomorphism invariance condition, along with the $E_\kappa$ for $\kappa$ countable. As just remarked, the finite ones are recovered once one includes the identity $I$. The quantifier"there exist infinitely many", for $\kappa = \aleph_0$ is a borderline case to which intuition and experience do not provide a clearcut answer as to its status. It can, however, be assimilated to logical notions under the homomorphism invariance criterion simply by restricting one's consideration to those operations which are invariant over infinite domains $M_0$, without thereby including the $E_\kappa$ for $\kappa$ uncountable. The"completeness" argument for logicality (suggested by Quine in the case of $=$) here gives quite anomalous results, since one has a complete logic for $E_\kappa$ for the case that $\kappa = \aleph_1$ by the work of Keisler (1970) while, as is well known, there is no such logic for the case that $\kappa = \aleph_0$.

I also agree with Quine (1986, pp. 64 ff) that second-order and higher-order quantification go beyond the bounds of logic. He takes these (famously) to be "set theory in sheep's clothing", and it is certainly true that the understood *meaning* of such quantifiers depends on what sets exist, or alternatively —if such quantifiers are regarded as binding predicate variables—of what predicates exist. To put it in other more explicit terms, that dependence (on what objects exist) certainly holds if second-order logic is to be more than a two-sorted version of first-order logic and is taken to verify the principle of existential instantiation in the form $\exists X \varphi(X) \rightarrow \varphi(\{x : \psi(x)\})$, which incorporates the comprehension principle for sets or predicates. But logic is supposed to be independent of ontology, and on those grounds, must exclude second-order and higher quantification. This is evidently a matter of some philosophical controversy, represented by the articles in the first part of the collection (Shapiro 1996), including a defense of second-order logic by Boolos (1975).[10] In any case, I count it as an argument in favor of the homomorphism invariance condition for logicality that it excludes second-order, and thence higher-order, quantification, by example 3) of the preceding section.

[10] I shall take up Boolos' proposed "first-orderization" of second-order logic via plural quantification in Sec. 6.

## 5.3 Generalized quantifiers, logical operations and natural language.

Mostowski (1957) introduced generalized quantifiers as follows (with change of notation to accord with that used here). A quantifier over a domain $M_0$ is a function $Q^M : \wp(M_0) \to \{T, F\}$ which is invariant under permutations of that domain; given this invariance condition, it can alternatively be identified with the collection $\{A \subseteq M_0 | Q^M(A) = T\}$. A quantifier in general is a function $Q$ from domains $M_0$ to quantifiers $Q^M$ on $M_0$. Quantifiers in this sense correspond to operations $O$ of type $(\pi \to b)$, i.e. $((0 \to b) \to b)$, which are invariant under the similarity relations $M \sim M$ given by a permutation of the underlying domain.

Lindström (1966) generalized considerably the notion of generalized quantifier, as follows. By a relational signature is meant a sequence of natural numbers $\overline{k} = (k_1, \dots, k_n)$. Then a quantifier of signature $\overline{k}$ is a collection $\mathbf{K}$ of relational structures $(M_0, R_1, \dots, R_n)$, of signature $\overline{k}$, i.e. where each $R_i$ is a $k_i$-ary relation on $M_0$, such that $\mathbf{K}$ is closed under isomorphism. (0-ary relations are identified with the truth values $T$ or $F$.)

Alternatively, a quantifier in Lindström's sense can be identified with a function $Q$ which assigns to each domain $M_0$ a function

$$Q^M : \wp(M_0^{k_1}) \times \dots \times \wp(M_0^{k_n}) \to \{T, F\},$$

with $Q^M(R_1, \dots, R_n) = T$ iff $(M_0, R_1, \dots, R_n) \in \mathbf{K}$. Such $Q$ may be identified with operations $O$ of type $((0^{k_1} \to b), \dots, (0^{k_n} \to b) \to b)$ which are invariant under isomorphisms $M \cong M'$.

The role of generalized quantifiers in natural language (NL) goes back to Montague (1970), though not explicitly connected with either of the preceding. Barwise and Cooper (1981) made the first systematic study of quantifiers in NL in relation to generalized quantifiers in Mostowski's sense. They identified NL quantifiers with noun phrases (NPs) which are, as usual, followed by verb phrases (VPs) to form sentences (Ss); one way to form a NP is as a Determiner (Det) followed by a common noun (N). From a semantic point of view items in both N and VP are interpreted as predicates, i.e. unary relations, and a Det acts like a Lindström quantifier of signature (1,1).

Keenan and Westerståhl (1996) ([K-W] in the following) provide an up-to-date survey of work on generalized quantifiers in NL, with a substantial list of references, and my discussion of the question of which operations should be counted as logical as supported by NL evidence is based on that presentation.

Basically three kinds of quantifiers $Q$ are initially considered in [K-W], denoted by them to be of type $\langle 1\rangle, \langle 1,1\rangle$ and $\langle\langle 1,1\rangle,1\rangle$, resp. Over each domain $M_0$, a $Q$ of type $\langle 1\rangle$ acts like a function $Q^M : \wp(M_0) \to \{T, F\}$; typical examples in NL are proper nouns, pronouns, and indefinite pronouns. We content ourselves with one example for purposes below: '*Some* like it hot' ; refer to [K-W] for many more examples. Quantifiers of type $\langle 1,1\rangle$ include one-place Dets which combine with common nouns to form type $\langle 1\rangle$ expressions; semantically, they map $\wp(M_0)$ into type $\langle 1\rangle$ objects. Here one has, among the many, many examples, '*All poets* daydream', and '*Most linguists* are bilingual'. Finally, the two-placed determiners such as *more...than...* in '*More* students *than* teachers attended the party', are counted as quantifiers of type $\langle\langle 1,1\rangle,1\rangle$; semantically, they are treated as maps from $\wp(M_0) \times \wp(M_0)$into type $\langle 1\rangle$. To relate to our notation using $\pi = (0 \to b)$, the quantifiers of these three types match up with operations of type $(\pi \to b)$, $(\pi \to (\pi \to b))$, and $(\pi^2 \to (\pi \to b))$, resp.; and the second and third of these are in correspondence with operations of type $(\pi^2 \to b)$ and $(\pi^3 \to b)$, resp. All of these are thus *monadic* operations, and fall under Lindström quantifiers of signature (1,1,...,1). We shall consider *polyadic* operations in NL separately, below. My first concern here is how logicality of quantifiers relates to NL evidence. The criterion taken in [K-W], pp. 849 ff. is, "standardly", isomorphism invariance. This would admit all the Mostowski quantifiers in type $\langle 1\rangle$, though most of those do not occur in everyday NL usage. [K-W], p. 851, extend logicality to cover also vague cardinality quantifiers, such as *about a hundred* in '*About a hundred* students are in my physics class', on the grounds that this can be taken to satisfy the isomorphism invariance condition by specification according to context. But they go on (p. 852) to point out that if sensitivity to context is permitted, the isomorphism invariance condition permits implausible Dets to act as logical operations, e.g. a Det $D$ which is interpreted as *every* in domains of less than 10 elements and as *some* when there are $\geq 10$ elements in the domain. This leads to my second concern, which was the main point in my critique of the Tarski-Sher thesis, that it does not explain the idea of same logical operation across all domains. That is addressed by [K-W], pp. 854-855 in terms of a condition called *extension* (EXT) suggested by van Benthem for type $\langle 1,1\rangle$ quantifiers $Q$, which is met if $Q^{M'}$ is an extension of $Q^M$ whenever $M_0 \subseteq M_0'$. The constraint EXT is an apparent NL universal, as witnessed for example by the $\forall$ quantifier considered as a type $\langle 1,1\rangle$ Det in $\forall AB$, interpreted as '*Every* A is a B', which is independent of the universe encompassing the As and the Bs (classes of supposedly fixed extent). But [K-W] acknowledge (p. 856, ftn. 10) that the corresponding condition for type $\langle 1 \rangle$ quantifiers fails, as witnessed, again by $\forall$ in the form $\forall A$, interpreted as '*Everything* is an $A$'. It is here, in my opinion, that homomorphism invariance wins the day as a criterion for logicality which explains sameness across domains of different sizes.

### 5.4 Monadic vs. polyadic quantification.

Next comes the question whether NL evidence supports the view of quantifiers as, first and foremost, monadic operators. Examples such as '*Most critics* reviewed *just four films*', seem to contravene this. [K-W], pp. 867 ff., construe this as a quantifier $Q$ of type $\langle\langle 1,1\rangle, 2\rangle$ which acts to take monadic predicates $A$, $B$ (*critic, film*) and a binary relation $R$ (*reviewed*) to $Q(A,B)(R)$, whose semantics is:

$$Q(A,B)(R) = T \iff$$
$$|\{a \in A : |\{b \in B : R(a,b)\}| = 4\} > |\{a \in A : |\{b \in B : R(a,b)\}| \neq 4\}.$$

If construed as a Lindström quantifier $\mathbf{K}$ of type (1, 1, 2), this is essentially polyadic. But it is shown in [K-W] how to interpret the given example as a kind of composition of two type $\langle 1 \rangle$ quantifiers, *most critics* and *(just) four films*, obtained from the type $\langle 1,1 \rangle$ quantifiers *most* and *four* with the relation reviewed, as

*most critics (four films (reviewed)).*

This saves treating the quantifiers involved as monadic, at the price of "lifting" the type by two abstraction steps with respect to an argument. The form is

$$Q_1(A)(\{a \in A : Q_2(B)(\{b : R(a,b)\})\}).$$

Thus both $Q_1$ and $Q_2$ remain of type $(\pi \to (\pi \to b))$, and are in turn themselves obtained by abstraction from operations $O_1$ and $O_2$ of type $(\pi^2 \to b)$. The entire second half of [K-W] is devoted to the general question of reducibility of polyadic quantifiers in NL to monadic quantifiers via suitable lifting by abstraction, and they come to the following generalization:

> Polyadic quantification in natural languages in general results from lifting monadic quantifiers. ([K-W], p. 890).

The NP-VP analysis of NL sentences shows the centrality of monadic predicates to human thought as expressed in natural language. I take the preceding conclusion in [K-W], together with the ubiquity of monadic quantifiers of the sort considered in 5.3, as evidence that the same extends to quantifiers in general. Combined with my argument for homomorphism invariance as a criterion for sameness of operations across domains, this makes it plausible that the class of operations definable from homomorphism invariant *monadic* operations is a natural one to consider from the point of view of what one might call "natural logic". If that is granted, then Theorem 6 supports my view that the first-order predicate calculus PC enjoys a privileged role in human thought.

We have seen in example 7) of Sec. 4 that homomorphism invariance when applied to essentially polyadic quantifiers, like that for well-foundedness of a relation, takes one out of the PC definable operations. As a purely theoretical question then, that leads us to the following:

**Question.** Is there a natural characterization of the homomorphism invariant propositional operations in general, in terms of logics extending PC?

## 5.5 Other approaches?

In the introduction I pointed out that a basic methodological choice has to be made when considering the question: what is a logical operation? Just to give a brief indication, the following are two quite different approaches, methodologically, from the model-theoretic ones studied in this paper. They will not be examined here, but would need to be dealt with in any full-scale comparative analysis of the general problem.

**(a) Proof-theoretic.** The idea here goes back to the work of Gentzen and Prawitz on systems of natural deduction, namely that the meaning of a logical operation is given by its rules of introduction. When that is explained in precise formal terms, one can then try to see which operations it is possible to characterize in that way. Relevant publications of importance here are (Zucker and Tragesser 1978), (Zucker 1978), and (Hacking 1979); cf. also (Došen 1994) for a general discussion and further references. The first of these leads exactly to PC.

**(b) Holistic.** By this I mean model-theoretic characterizations of logics as a whole, without attempting to isolate the separate contributions of individual operations which may generate them. The first and most famous example of such is provided by (Lindström 1969), which characterizes first-order logic *with* equality PC(=) in terms of general properties of its satisfaction relation. That work was paradigmatic for considerable further research, surveyed in many parts of (Barwise and Feferman 1985). Tharp (1975) contains an interesting discussion.

# 6 Tarski and Boolos on logicism.

In his "What are logical notions?" lecture that was the starting point for this paper, Tarski concluded with a discussion of its relevance to the logicist program, as follows:

> The question is often asked whether mathematics is a part of logic. Here we are interested in only one aspect of this problem, whether mathematical notions are logical notions, and not, for example, in whether mathematical truths are logical truths, which is outside our domain of discussion. (Tarski 1986, p. 151)

His answer is, curiously: "As you wish"! The argument is that since "the whole of mathematics can be constructed within set theory, or the theory of classes", and since "all usual set-theoretical notions" can be defined in terms of the relation of membership, the determination comes down to whether membership is a logical notion. [11] But—Tarski goes on—two methods have been provided for the foundations of set theory following the discovery of paradoxes in that subject, namely the theory of types as exemplified in *Principia Mathematica* (which he takes implicitly in unramified form), and axiomatic set theory as formulated by Zermelo, et al. If one follows the method of the theory of types then membership *is* a part of logic, since it is invariant under the extension to higher types of any permutation of the domain of individuals. On the other hand, if axiomatic set theory is followed, there is "only one universe of discourse and the membership relation between its

[11] It is also curious that Tarski ignores the fact due to his fundamental result on the non-definability of truth-in-$L$ within a language $L$, that the mathematical notion of truth of sentences of the language of set theory cannot be defined within set theory (and similarly for type theory).

individuals is an undefined relation, a primitive notion." On that account, membership is *not* a logical notion, since as Tarski had shown earlier, there are only four permutation-invariant relations between individuals, the universal relation, the empty relation, the identity relation and its complement. Tarski winds up these considerations as follows:

> This conclusion ["As you wish!"] is interesting, it seems to me, because the two possible answers correspond to two different types of mind. A monistic conception of logic, set theory, and mathematics, where the whole of mathematics would be a part of logic, appeals, I think, to a fundamental tendency of modern philosophers. Mathematicians, on the other hand, would be disappointed to hear that mathematics, which they consider the highest discipline in the world, is a part of something so trivial as logic; and they therefore prefer a development of set theory in which set-theoretical notions are not logical notions. The suggestion which I have made does not, by itself, imply any answer to the question of whether mathematical notions are logical. (Tarski 1986, p. 153)

Though Tarski's consideration only of the question "whether mathematical notions are logical notions" and not of "whether mathematical truths are logical truths" appears at first sight to be a reasonable one, it is not clear that the two can be separated so neatly. For, any argument one way or the other about the first question must necessarily invoke assumptions about various properties of the notions involved, and those lead one into the second question. Tarski skirts this by only considering the outer syntactic form of the two theories for "mathematics" that he compares, namely the theory of types and the theory of sets. Moreover, he treats these questions in an all-or-nothing way, not distinguishing whether some prima-facie mathematical notions (or truths), such as those of arithmetic, might be logical, but not others, such as those of higher set theory. Indeed, on his thesis, it would appear that at least finite cardinal arithmetic is in a suitable sense a part of logic. Finally, what is puzzling to me about his whole discussion is that (as I have argued in Sec. 3) Tarski's thesis assimilates logic to a substantial part of set-theoretical mathematics, so that there is a circularity involved in the question whether mathematical notions are logical. But I will have to leave his views of the matter at that.

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Let me turn, finally, to Boolos' work on logicism, more specifically on Frege's program, to the study of which he contributed a number of incisive essays. I divide that into two parts, the first (a) being his defense, on nominalistic grounds via plural quantification, of the second-order logic that Frege used for his program, and the second (b) being his rejection of Frege's claim to having reduced arithmetic to logic, even after his system is reconstructed in a consistent form.

**(a) Sublogicism, second-order logic and plural quantification.** In the *Begriffsschrift*, Frege argued against Kant, as to the necessity of a priori intuitions for mathematics, that there are examples of mathematical judgments that "at first sight appear to be possible only on the basis of some intuition," but which can be brought forth in pure thought "solely from the content that results from its own constitution." In his "Reading the *Begriffsschrift*" (Boolos 1985a), Boolos defends a weakened version of Frege's logicism that he calls "*sublogicism* : the claim that there are (many) interesting mathematical truths that can be reduced (in the appropriate sense) to logic." (Boolos 1985a in his 1998, p. 156). This depends first of all on a careful reading of Frege's work and secondly on Boolos' claimed nominalistic reduction to plural quantification of the second-order quantification in the *Begriffsschrift*. It is the latter that is my main concern here. That aim was promoted in two papers, "To be is to be a value of a variable (or to be some values of some variables)" (Boolos 1984), and "Nominalist platonism" (Boolos 1985), both reprinted in (Boolos 1998). These essays have been given serious attention by a number of philosophers. What Boolos' claim to have "tamed" second-order logic in this way comes to is well summarized by Michael Resnik in his critique "Second-order logic still wild" (Resnik 1988), and there is no point in my repeating it here. His conclusion, with which I concur completely, is that no genuine ontological reduction is obtained thereby. "Boolos is involved in a circle: he uses second-order quantification to explain English plural quantification and uses this, in turn, to explain second-order quantification." (Resnik 1988, p. 83).[12] Whatever the merits of this proposed reduction, even Boolos

[12] Another critique which has been brought to my attention is that of (Shapiro 1993). On the other hand, as Geoffrey Hellman has written me, David Lewis (1991) 62–71 makes (what he considers to be ) a persuasive case that we *do* have an independent grasp of plural quantification that doesn't have to be explained in terms of second-order quantifi-

does not accept Frege's logicism in full on its basis, and the reasons for that are what I turn to next.

**(b) The consistent reconstruction of Frege's arithmetic, and its significance.** In his paper "The consistency of Frege's Foundations of Arithmetic" (Boolos 1987, reprinted in Boolos 1998), a second-order system FA is presented which is shown to be consistent, unlike that of Frege's *Grundlagen der Arithmetik*, in which the mathematical results of the latter can be redeveloped exactly following Frege's work. The system FA has three sorts of variables: first-order (individual or object) variables $x, y, z, \ldots$; unary second-order (or "concept") variables $F, G, H, \ldots$; and binary second-order (relation) variables $\varphi, \psi \ldots$. There is only one nonlogical symbol, $\eta$, relating concept and object variables. The atomic formulas are of the form $Fx$, $x\varphi y$ and $F\eta x$. Equality is defined by: $x = y \longleftrightarrow \forall F(Fx \longleftrightarrow Fy)$. Using the binary second-order variables, the equinumerosity relation, $F\ eq\ G$, between two concepts $F$, $G$ is defined as usual. There are three main axioms. The first two are the usual comprehension axiom schemes consisting of the universal closures of

$$\exists F \forall x(Fx \longleftrightarrow A(x)),$$

where $A$ is a formula that does not contain '$F$' free, and

$$\exists \varphi \forall x \forall y(x\varphi y \longleftrightarrow B(x, y)),$$

where $B$ is a formula that does not contain '$\varphi$' free. Boolos regards both of these as logical. The third (nonlogical) axiom is the single sentence

**Numbers:** $\forall F \exists ! x \forall G(G\eta x \longleftrightarrow F\ eq\ G).$

The idea is that the unique $x$ associated with $F$ by this axiom is a first-order object representing the equivalence class of $F$ under the relation of equinumerosity. If we denote that unique $x$ by $N(F)$, one has, as a consequence of Numbers,

**Hume's principle:** $\forall F \forall G(N(F) = N(G) \longleftrightarrow F\ eq\ G).$

cation, though there appears to be an asymmetry between existential plurals (natural) and universal plurals (not natural) in English.

Frege had used axioms for extensions of concepts to define the operation $N$ in such a way as to be able to derive Hume's principle in Section 73 of his *Grundlagen*. Russell's paradox showed that those axioms are inconsistent. By contrast, Boolos shows that FA, which does not use extensions, is consistent. Then, following the lead of Wright (1983), he shows that "(o)nce Hume's principle is proved, *Frege makes no further use of extensions*." (Boolos 1998, p. 191). In his discussion of the significance of this work, Boolos comes to the following provocative conclusions (op. cit., p. 200): "(1) Numbers is no logical truth; and therefore (2) Frege did not demonstrate the truth of logicism in the *Foundations of Arithmetic*. (3) Logic is synthetic if mathematics is, because (4) there are many interesting, logically true conditionals with antecedent Numbers whose mathematical content is not appreciably less than that of their consequents." And he adds to these: "(5) Since we have no understanding of the role of logic or mathematics in cognition, the failure of logicism is at present quite without significance for our understanding of mentality." In view of my working identification of logic with the first-order predicate calculus PC, I am in agreement with (1) and (2). I am more or less in disagreement with (3), though I don't have strong feelings about what being synthetic amounts to. I don't see (4) since all results of mathematics can be represented as logical consequences of mathematical hypotheses. As to (5), I agree with the conclusion, but not the premise; it seems to me that we *do* have *some* understanding of the role of logic, and to some extent of mathematics, in cognition, though we surely have much farther to go in both respects. To reiterate my introductory remarks, I think that the theoretical study of what a logical operation is, and hence of what the scope of logic is, must be connected with the more empirical study of the role of logic in the exercise of human rationality. I am optimistic that a better understanding of either will inform the other.

Stanford University
sf@csli.stanford.edu

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