

A NEW APPROACH TO COMPUTE THE ALTERNATING POWER SUMS VIA BROUSSEAU SUMMATION

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Summary. In this article, we present a new explicit formula for calculating the weighted Brousseau sums associated with the geometric sequence, utilizing only binomial coefficients, as long as the common ratio is not equal to one. We use it to derive the polynomial forms of the alternating power sums. Furthermore, we deduce expressions for the Brousseau sums of Mersenne, Jacobsthal, and Jacobsthal-Lucas numbers. We also derive a recursive formula for finding the power sums.

1 INTRODUCTION

Jacques Bernoulli [1, p. 95-97] is credited with introducing the Bernoulli numbers, which are denoted as B_r , to calculate the power sum $\sum_{k=1}^n k^p$ of the p^{th} powers of the first n positive integers. Subsequently, he established the summation formula

$$\sum_{k=1}^n k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{p+1} \sum_{j=1}^{\lfloor p/2 \rfloor} \binom{p+1}{2j} (-1)^{j+1} B_{2j} n^{p-2j+1},$$

for $m, n \geq 1$.

By defining $B_0 = 1$ and $B_1 = -\frac{1}{2}$, it can be written as (see [2])

$$\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{r=0}^p (-1)^r \binom{p+1}{r} B_r n^{p-r+1}, \quad (1)$$

for $m, n \geq 1$.

Euler [3] investigated the alternating power sum $\sum_{k=1}^n (-1)^k k^p$ two decades later. In the end, he provided the general conclusion [4, Identity (2.8)]

$$\sum_{k=1}^n (-1)^k k^p = \sum_{j=1}^p (-1)^{+j} \binom{p}{j} \frac{t^{n+1} n^j}{(t-1)^{-j+1}} A_{-j}(t) + (-1)^{-} \frac{t(t^n-1)}{(t-1)^{+1}} A_{+1}(t), \quad (2)$$

where $A_{\pm r}(t)$ are the *Eulerian polynomials*, which are recursively defined by

$$A_0(t) = 1, \quad A_{\pm r}(t) = \sum_{r=0}^{-1} \binom{p}{r} A_r(t) (t-1)^{-r-1}.$$

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Euler [4, Identity (3.3)] provided another simplified version of Eq. (2) as

$$\sum_{k=1}^n k^p t^k = -t^{n+1} \sum_{j=0}^p \binom{p}{j} \frac{n^{p-j}}{(1-t)^{j+1}} A_j(t) + \frac{t}{(1-t)^{p+1}} A_p(t). \quad (3)$$

In this paper, we provide a better version of this formula for the weighted power sum. It is derived exclusively from binomial coefficients as

$$\sum_{k=1}^n k^p t^k = \frac{t}{t-1} \left[\left(\sum_{r=0}^p \binom{p}{r} (-1)^r \mathcal{A}^{(r)}(t) n^{p-r} \right) t^n - (-1)^p \mathcal{A}^{(p)}(t) \right],$$

where the rational functions $\mathcal{A}^{(p)}(t)$ are recursively defined by

$$\mathcal{A}^{(p)}(t) = \begin{cases} 1 & , \text{ if } p = 0 ; \\ \frac{1}{t-1} \sum_{j=1}^p \binom{p}{j} \mathcal{A}^{(p-j)}(t), & \text{ if } p \geq 1 . \end{cases}$$

The *Euler polynomials* $E_p(t)$ are defined by the explicit formula [5, p. 804]

$$E_p(x) = \sum_{j=0}^p \binom{p}{j} \frac{E_j}{2^j} \left(x - \frac{1}{2}\right)^{p-j},$$

where E_j are the Euler numbers. The Euler numbers are connected to a specific value of the Euler polynomials, which is given by the formula:

$$E_p = 2^p E_p\left(\frac{1}{2}\right).$$

The alternating power sum can be written in terms of Euler polynomials as [5, p. 804]

$$\sum_{k=1}^n (-1)^k k^p = \frac{1}{2} [(-1)^n E_p(n+1) + E_p(0)]. \quad (4)$$

Equivalently, Kim [6, Theorem 1] proved that

$$\sum_{k=0}^{n-1} (-1)^k k^p = \frac{(-1)^{n+1}}{2} \sum_{j=0}^{p-1} \binom{p}{j} E_j(0) n^{p-j} + \frac{E_p(0)}{2} (1 + (-1)^{n+1}). \quad (5)$$

Additional information regarding alternating power sums is available in [7, 8, 9, 10, 11].

Euler's identity in Eq. (4) gives the alternating power sums in terms of powers of $(n+1)$. We deduce a simplified version in terms of powers of n , which is given below.

$$\sum_{k=1}^n (-1)^k k^p = \frac{1}{2} [(-1)^n \bar{\mathcal{E}}^{(p)}(n) - \bar{\mathcal{E}}^{(p)}(0)],$$

where

$$\bar{\mathcal{E}}^{(p)}(x) = \sum_{r=0}^p (-1)^r \mathcal{E}^{(r)} \binom{p}{r} x^{p-r},$$

are the shifted Euler polynomials and $\mathcal{E}^{(r)} = \mathcal{A}^{(r)}(-1)$. In fact, $\mathcal{A}^{(r)}(-1)$ is the value of the r^{th} Euler polynomial $E_r(x)$ evaluated at $x = 0$.

Furthermore, we deduce identities for the weighted Brousseau sums

$$\sum_{k=1}^n k^p M_k, \quad \sum_{k=1}^n k^p J_k, \quad \text{and} \quad \sum_{k=1}^n k^p j_k,$$

where, the Mersenne numbers M_k , the Jacobsthal numbers J_k , and the Jacobsthal-Lucas numbers j_k are defined by [12, p. 447]

$$M_k = 2^k - 1, \quad J_k = \frac{1}{3}(2^k - (-1)^k), \quad \text{and} \quad j_k = 2^k + (-1)^k,$$

for $k \geq 1$.

2 RECURSIONS

In this section, we find a recurrence formula for the convolution $\sum_{k=0}^{n-1} k^p t^{n-k}$. Recall that the geometric sequence $(G_n)_{n \geq 0}$ is defined by $G_n = t^n$. We can express it in recursive form as $G_0 = 1$ and $G_n = tG_{n-1}$ for $n > 0$.

Lemma 1. *For all integers $n, p \geq 1$, the following identity holds:*

$$G_n - tn^p = \sum_{k=1}^{n-1} \left(k^p t - \sum_{j=0}^p \binom{p}{j} k^{p-j} \right) G_{n-k}. \quad (6)$$

Proof. Consider the right-hand side of Eq. (6) and use the binomial expansion to get,

$$\begin{aligned} \sum_{k=1}^{n-1} \left(k^p t - \sum_{j=0}^p \binom{p}{j} k^{p-j} \right) G_{n-k} &= \sum_{k=1}^{n-1} (k^p t - (k+1)^p) t^{n-k} \\ &= \sum_{k=1}^{n-1} k^p t^{n-k+1} - \sum_{k=1}^{n-1} (k+1)^p t^{n-k} \\ &= \sum_{k=1}^{n-1} k^p t^{n-k+1} - \sum_{k=2}^n k^p t^{n-k+1} \\ &= t^n - tn^p \\ &= G_n - tn^p. \end{aligned}$$

This completes the proof. □

Proposition 2. *For all integers $n, p \geq 1$,*

$$\sum_{k=0}^{n-1} k^p G_{n-k} = \frac{1}{t-1} \left[\sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} G_{n-k} \right) - tn^p \right]. \quad (7)$$

Proof. Rewriting Eq. (6), we have

$$\begin{aligned} G_n &= tn^p + \sum_{k=1}^{n-1} \left(k^p(t-1) - \sum_{j=1}^p \binom{p}{j} k^{p-j} \right) G_{n-k} \\ &= tn^p + (t-1) \sum_{k=1}^{n-1} k^p G_{n-k} - \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=1}^{n-1} k^{p-j} G_{n-k} \right). \end{aligned}$$

Since $k^p G_{n-k} = 0$ for $k = 0$, we may begin the first summation at $k = 0$ instead of at $k = 1$ and obtain

$$G_n = tn^p + (t-1) \sum_{k=0}^{n-1} k^p G_{n-k} - \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=1}^{n-1} k^{p-j} G_{n-k} \right). \quad (8)$$

Since

$$k^{p-j} G_{n-k} = \begin{cases} 0 & \text{if } k = 0, j \neq p, \\ G_n & \text{if } k = 0, j = p, \end{cases}$$

we can rewrite (8) as

$$G_n = tn^p + (t-1) \sum_{k=0}^{n-1} k^p G_{n-k} - \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} G_{n-k} \right) + G_n.$$

Thus, we obtain

$$(t-1) \sum_{k=0}^{n-1} k^p G_{n-k} = \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} G_{n-k} \right) - tn^p.$$

This completes the proof. \square

Let us explore a case that utilizes Proposition 2. It follows from Eq. (7) that

$$\sum_{k=0}^{n-1} k^p t^{n-k} = \frac{1}{t-1} \left[\sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} t^{n-k} \right) - tn^p \right],$$

and

$$\sum_{k=0}^{n-1} k^p = \lim_{t \rightarrow 1} \sum_{k=0}^{n-1} k^p t^{n-k}$$

$$= \lim_{t \rightarrow 1} \frac{1}{t-1} \left[\sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} t^{n-k} \right) - t n^p \right]$$

Now, using L'Hôpital's rule, we obtain

$$\begin{aligned} \sum_{k=0}^{n-1} k^p &= \lim_{t \rightarrow 1} \left[\sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} (n-k) t^{n-k-1} \right) - n^p \right] \\ &= \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} (n-k) k^{p-j} \right) - n^p \\ &= n \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} \right) - \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j+1} \right) - n^p \\ &= n \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} \right) - \sum_{j=2}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j+1} \right) - p \sum_{k=0}^{n-1} k^p - n^p. \end{aligned}$$

Thus,

$$(p+1) \sum_{k=0}^{n-1} k^p = n \sum_{j=1}^p \binom{p}{j} \left(\sum_{k=0}^{n-1} k^{p-j} \right) - \sum_{j=1}^{p-1} \binom{p}{j+1} \left(\sum_{k=0}^{n-1} k^{p-j} \right) - n^p.$$

Replacing n by $n+1$ and defining $\binom{p}{p+1} = 0$, we have the following novel recurrence for computing the power sum:

Theorem 3. For all integers $n, p \geq 1$,

$$(p+1) \sum_{k=0}^n k^p = \sum_{j=1}^p \left[(n+1) \binom{p}{j} - \binom{p}{j+1} \right] \left(\sum_{k=0}^n k^{p-j} \right) - (n+1)^p. \quad (9)$$

Setting $p = 1$ yields

$$2 \cdot \sum_{k=0}^n k = (n+1) \left(\sum_{k=0}^n 1 \right) - (n+1) = (n+1)^2 - (n+1) = n(n+1).$$

Thus,

$$\sum_{k=0}^n k = \frac{n(n+1)}{2}.$$

Likewise, setting $p = 2$ yields

$$\begin{aligned}
 3 \cdot \sum_{k=0}^n k^2 &= [2(n+1) - 1] \binom{n}{k=0} + [(n+1) - 0] \binom{n}{k=0} - (n+1)^2 \\
 &= (2n+1) \frac{n(n+1)}{2} + (n+1)^2 - (n+1)^2 \\
 &= \frac{n(n+1)(2n+1)}{2}.
 \end{aligned}$$

Thus,

$$\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3 CONVOLUTIONS

In this section, we find an explicit formula for the convolution $\sum_{k=0}^{n-1} k^p t^{n-k}$. Using Proposition 2, we can find the sum

$$T_n^{(p)}(t) = \sum_{k=0}^{n-1} k^p G_{n-k}$$

recursively for $p = 1, 2, \dots$

$$(t-1)T_n^{(0)}(t) = G_{n+1} - t,$$

$$(t-1)T_n^{(1)}(t) = \left(\frac{1}{t-1}\right) G_{n+1} - t \left[n + \frac{1}{t-1}\right],$$

$$(t-1)T_n^{(2)}(t) = \left(\frac{t+1}{(t-1)^2}\right) G_{n+1} - t \left[n^2 + 2\left(\frac{1}{t-1}\right)n + \frac{t+1}{(t-1)^2}\right],$$

$$\begin{aligned}
 (t-1)T_n^{(3)}(t) &= \left(\frac{t^2+4t+1}{(t-1)^3}\right) G_{n+1} \\
 &\quad - t \left[n^3 + 3\left(\frac{1}{t-1}\right)n^2 + 3\left(\frac{t+1}{(t-1)^2}\right)n + \frac{t^2+4t+1}{(t-1)^3}\right],
 \end{aligned}$$

and so on. These equations follow a pattern. To understand the pattern, we must define the following sequence $(\mathcal{A}^{(p)}(t))_{p \geq 0}$ of Eulerian rational functions:

$$\mathcal{A}^{(p)}(t) = \begin{cases} 1, & \text{if } p = 0, \\ \frac{1}{t-1} \sum_{j=1}^p \binom{p}{j} \mathcal{A}^{(p-j)}(t), & \text{if } p \geq 1. \end{cases} \quad (10)$$

The recurrence $\mathcal{A}^{(p)}(t)$ generates the sequence of rational functions

$$1, \frac{1}{t-1}, \frac{t+1}{(t-1)^2}, \frac{t^2+4t+1}{(t-1)^3}, \frac{t^3+11t^2+11t+1}{(t-1)^4}, \frac{t^4+26t^3+66t^2+26t+1}{(t-1)^5}, \dots$$

Note that the numerators are the Eulerian polynomials $A_p(t)$.

For $t = -1$, the recurrence generates the rational sequence

$$1, -\frac{1}{2}, 0, \frac{1}{4}, 0, -\frac{1}{2}, 0, \frac{17}{8}, 0, -\frac{31}{2}, 0, \frac{691}{4}, 0, -\frac{5461}{2}, \dots$$

Note that these numbers are the values of the Euler polynomials $E_p(t)$ evaluated at $t = 0$. For $t = 2$, this sequence is

$$1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, \dots$$

These numbers are known as Fubini numbers or ordered Bell numbers [13, A000670]. For notational convenience, we write

$$\mathcal{E}^{(p)} = E_p(0) = \mathcal{A}^{(p)}(-1) \quad \text{and} \quad \mathcal{F}^{(p)} = \mathcal{A}^{(p)}(2).$$

Thus,

$$\mathcal{E}^{(p)} = \begin{cases} 1 & , \text{ if } p = 0; \\ -\frac{1}{2} \sum_{j=1}^p \binom{p}{j} \mathcal{E}^{(p-j)}, & \text{ if } p \geq 1, \end{cases}$$

and

$$\mathcal{F}^{(p)} = \begin{cases} 1 & , \text{ if } p = 0; \\ \sum_{j=1}^p \binom{p}{j} \mathcal{F}^{(p-j)}, & \text{ if } p \geq 1. \end{cases}$$

It is easy to prove that

$$(-1)^p \mathcal{E}^{(p)} = \sum_{j=0}^p \binom{p}{j} \mathcal{E}^{(j)}.$$

The general formula for the convolution $T_n^{(p)}(t)$ is given here.

Theorem 4. For all integers $p \geq 0$,

$$\sum_{k=0}^{n-1} k^p G_{n-k} = \frac{1}{t-1} \left[\mathcal{A}^{(p)}(t) G_{n+1} - t \sum_{r=0}^p \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r} \right]. \quad (11)$$

In other words,

$$\sum_{k=0}^{n-1} k^p t^{n-k} = \frac{t}{t-1} \left[\mathcal{A}^{(p)}(t) t^n - \sum_{r=0}^p \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r} \right].$$

Proof. We use induction on p . When $p = 0$, the left-hand side of Eq. (11) is

$$\sum_{k=0}^{n-1} k^0 G_{n-k} = \sum_{k=0}^{n-1} G_{n-k} = \sum_{k=1}^n G_k = \frac{G_{n+1} - t}{t-1},$$

and the right-hand side is

$$\frac{1}{t-1}(\mathcal{A}(0)G_{n+1} - t\mathcal{A}(0)) = \frac{G_{n+1} - t}{t-1}.$$

Thus, Eq. (11) holds for $p = 0$. Now fix $p \geq 1$. Assume that Eq. (11) holds for all non-negative integers less than p . Now, using Proposition 2, we have

$$(t-1) \sum_{k=0}^{n-1} k^p G_{n-k} = \sum_{j=1}^p \binom{p}{j} \left[\sum_{k=0}^{n-1} k^{p-j} G_{n-k} \right] - tn^p. \quad (11)$$

Now, using the induction hypothesis,

$$\sum_{k=0}^{n-1} k^{p-j} G_{n-k} = \frac{1}{t-1} \left[\mathcal{A}^{(p-j)} G_{n+1} - t \sum_{r=0}^{p-j} \mathcal{A}^{(r)} \binom{p-j}{r} n^{p-j-r} \right].$$

By substituting this in Eq. (11), we obtain

$$\begin{aligned} (t-1) \sum_{k=0}^{n-1} k^p G_{n-k} &= \sum_{j=1}^p \binom{p}{j} \left[\frac{1}{t-1} \left(\mathcal{A}^{(p-j)} G_{n+1} - t \sum_{r=0}^{p-j} \mathcal{A}^{(r)} \binom{p-j}{r} n^{p-j-r} \right) \right] - tn^p \\ &= \frac{1}{t-1} \left[\sum_{j=1}^p \binom{p}{j} \mathcal{A}^{(p-j)} G_{n+1} - t \sum_{j=1}^p \sum_{r=0}^{p-j} \mathcal{A}^{(r)} \binom{p}{j} \binom{p-j}{r} n^{p-j-r} \right] - tn^p. \end{aligned}$$

Using Eq. (10), this becomes

$$(t-1) \sum_{k=0}^{n-1} k^p G_{n-k} = \mathcal{A}^{(p)} G_{n+1} - t\phi^{(p)}(t, n) - tn^p, \quad (13)$$

where

$$\phi^{(p)}(t, n) = \frac{1}{t-1} \sum_{j=1}^p \sum_{r=0}^{p-j} \mathcal{A}^{(r)} \binom{p}{j} \binom{p-j}{r} n^{p-j-r}. \quad (14)$$

Keeping j as j and changing r to $r+j$ in Eq. (14), we obtain

$$\phi^{(p)}(t, n) = \frac{1}{t-1} \sum_{j=1}^p \sum_{r=j}^p \mathcal{A}^{(r-j)}(t) \binom{p}{j} \binom{p-j}{r-j} n^{p-r}.$$

Now, switching the order of summation, we get

$$\phi^{(p)}(t, n) = \frac{1}{t-1} \sum_{r=1}^p \sum_{j=1}^r \mathcal{A}^{(r-j)}(t) \binom{p}{j} \binom{p-j}{r-j} n^{p-r}.$$

Next, using the binomial identity [14, Identity 134, p. 67], we have

$$\binom{p}{j} \binom{p-j}{r-j} = \binom{p}{r} \binom{r}{j}.$$

Therefore,

$$\begin{aligned} \phi^{(p)}(t, n) &= \frac{1}{t-1} \sum_{r=1}^p \sum_{j=1}^r \mathcal{A}^{(r-j)}(t) \binom{p}{r} \binom{r}{j} n^{p-r} \\ &= \sum_{r=1}^p \binom{p}{r} n^{p-r} \left(\frac{1}{t-1} \sum_{j=1}^r \binom{r}{j} \mathcal{A}^{(r-j)}(t) \right) \\ &= \sum_{r=1}^p \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r} \\ &= \sum_{r=0}^p \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r} - n^p. \end{aligned}$$

Substituting this in Eq. (13) yields

$$\begin{aligned} (t-1) \sum_{k=0}^{n-1} k^m G_{n-k} &= \mathcal{A}^{(p)} G_{n+1} - t \left(\sum_{r=0}^p \mathcal{A}^{(r)} \binom{p}{r} n^{p-r} - n^p \right) - t n^p \\ &= \mathcal{A}^{(p)} G_{n+1} - t \left(\sum_{r=0}^p \mathcal{A}^{(r)} \binom{p}{r} n^{p-r} \right). \end{aligned}$$

Hence,

$$\sum_{k=0}^{n-1} k^m G_{n-k} = \frac{1}{t-1} \left[\mathcal{A}^{(p)}(t) G_{n+1} - t \left(\sum_{r=0}^p \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r} \right) \right],$$

as desired. This completes the proof. \square

Corollary 5. For all integers $p \geq 0$, the following identity holds:

$$\sum_{k=0}^{n-1} (-1)^{n-k} k^p = \frac{1}{2} \left[(-1)^n \mathcal{E}^{(p)} - \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} n^{p-r} \right]. \quad (15)$$

Proof. It follows from Theorem 4 by setting $t = -1$. \square

Corollary 6. For all integers $p \geq 0$, the following identity holds:

$$\sum_{k=0}^{n-1} 2^{n-k} k^p = \mathcal{F}^{(p)} 2^{n+1} - 2 \sum_{r=0}^p \mathcal{F}^{(r)} \binom{p}{r} n^{p-r}.$$

Proof. It follows from Theorem 4 by setting $t = 2$. \square

Definition 7. For all integers $p \geq 0$, we define the following polynomials $\mathcal{E}^{(p)}(x)$ in x of degree p as

$$\mathcal{E}^{(p)}(x) = \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} x^{p-r}.$$

Note that $\mathcal{E}^{(p)}(0) = \mathcal{E}^{(p)}$.

The first few of the polynomials $\mathcal{E}^{(p)}$ are given below:

$$\begin{aligned} \mathcal{E}^{(0)}(x) &= 1, & \mathcal{E}^{(1)}(x) &= x - \frac{1}{2}, \\ \mathcal{E}^{(2)}(x) &= x^2 - x, & \mathcal{E}^{(3)}(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \\ \mathcal{E}^{(4)}(x) &= x^4 - 2x^3 + x, & \mathcal{E}^{(5)}(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}, \\ \mathcal{E}^{(6)}(x) &= x^6 - 3x^5 + 5x^3 - 3x, & \mathcal{E}^{(7)}(x) &= x^7 - \frac{7}{2}x^6 + \frac{35}{4}x^4 - \frac{21}{2}x + \frac{17}{8}. \end{aligned}$$

Note that these are nothing but the *Euler polynomials* $E_p(x)$. It is not difficult to prove by using the exponential generating function that

$$E_p(x) = \mathcal{E}^{(p)}(x) = \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} x^{p-r}.$$

4 POWER SUMS

In this section, we find an explicit formula for the sum $\sum_{k=1}^n k^p t^n$ and deduce the formula for the alternating power sum. Applying Theorem 4, we can find the sums

$$S_n^{(p)}(t) = \sum_{k=1}^n k^p G_k$$

for $p = 0, 1, 2, \dots$

$$\begin{aligned} (t-1)S_n^{(0)}(t) &= G_{n+1} - t, \\ (t-1)S_n^{(1)}(t) &= \left(n - \frac{1}{t-1}\right)G_{n+1} + \frac{t}{t-1}, \\ (t-1)S_n^{(2)}(t) &= \left(n^2 - \frac{2}{t-1}n + \frac{t+1}{(t-1)^2}\right)G_{n+1} - \frac{t(t+1)}{(t-1)^2}, \\ (t-1)S_n^{(3)}(t) &= \left(n^3 - \frac{3}{t-1}n^2 + \frac{3(t+1)}{(t-1)^2}n - \frac{t^2+4t+1}{(t-1)^3}\right)G_{n+1} + \frac{t(t^2+4t+1)}{(t-1)^3}, \end{aligned}$$

and so on. These equations also follow a pattern. Each equation is of the form

$$(t-1)S_n^{(p)}(t) = \mathcal{C}^{(p)}(t, n)G_{n+1} - t\mathcal{C}^{(p)}(t, 0),$$

where

$$\mathcal{C}^{(p)}(t, n) = \sum_{r=0}^p (-1)^r \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r}.$$

Theorem 8. For all $p \geq 0$, we have

$$\sum_{k=1}^n k^p G_k = \frac{1}{t-1} \left[\left(\sum_{r=0}^p \binom{p}{r} (-1)^r \mathcal{A}^{(r)}(t) n^{p-r} \right) G_{n+1} - (-1)^p t \mathcal{A}^{(p)}(t) \right].$$

In other words,

$$\sum_{k=1}^n k^p t^k = \frac{t}{t-1} \left[\left(\sum_{r=0}^p \binom{p}{r} (-1)^r \mathcal{A}^{(r)}(t) n^{p-r} \right) t^n - (-1)^p \mathcal{A}^{(p)}(t) \right]. \quad (16)$$

Proof. The case where $p = 0$ holds since

$$\sum_{k=1}^n t^k = \frac{t^{n+1} - t}{t-1}.$$

Now, assume that $p \geq 1$. Then, using the binomial expansion, we have

$$\begin{aligned} \sum_{k=1}^n k^p G_k &= \sum_{k=0}^{n-1} (n-k)^p G_{n-k} \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=0}^p \binom{p}{j} n^{p-j} (-k)^j \right) G_{n-k} \\ &= \sum_{j=0}^p \sum_{k=0}^{n-1} \binom{p}{j} n^{p-j} (-1)^j k^j G_{n-k} \\ &= \sum_{j=0}^p (-1)^j \binom{p}{j} n^{p-j} \left(\sum_{k=0}^{n-1} k^j G_{n-k} \right). \end{aligned}$$

Thus,

$$\sum_{k=1}^n k^p G_k = \sum_{j=0}^p (-1)^j \binom{p}{j} n^{p-j} \left(\sum_{k=0}^{n-1} k^j G_{n-k} \right). \quad (17)$$

Now, using Theorem 4, we have

$$\sum_{k=0}^{n-1} k^j G_{n-k} = \frac{1}{t-1} \left[\mathcal{A}^{(j)}(t) G_{n+1} - t \sum_{r=0}^j \mathcal{A}^{(r)}(t) \binom{j}{r} n^{j-r} \right]. \quad (18)$$

When we substitute Eq. (18) into Eq. (17), we obtain

$$\sum_{k=1}^n k^p G_k = \frac{1}{t-1} \left[\left(\sum_{j=0}^m (-1)^j \mathcal{A}^{(j)} \binom{m}{j} n^{m-j} \right) G_{n+1} - t \sum_{j=0}^p \sum_{r=0}^j \mathcal{A}^{(r)}(t) (-1)^j \binom{m}{j} \binom{j}{r} n^{m-r} \right]$$

Thus,

$$\sum_{k=1}^n k^p G_k = \frac{1}{t-1} \left[\left(\sum_{j=0}^m (-1)^j \mathcal{A}^{(j)} \binom{m}{j} n^{m-j} \right) G_{n+1} - t \psi^{(p)}(t, n) \right],$$

where

$$\psi^{(p)}(t, n) = \sum_{j=0}^p \sum_{r=0}^j (-1)^j \mathcal{A}^{(r)}(t) \binom{p}{j} \binom{j}{r} n^{p-r}. \quad (19)$$

To complete the proof, we only need to show that

$$\psi^{(p)}(t, n) = (-1)^p \mathcal{A}^{(p)}.$$

By switching the order of summation in Eq. (19), we get

$$\psi^{(p)}(t, n) = \sum_{r=0}^p \mathcal{A}^{(r)}(t) \left(\sum_{j=r}^p (-1)^j \binom{p}{j} \binom{j}{r} n^{p-r} \right).$$

Now, using the binomial identity from Gould's collection [15, Identity 3.119],

$$\sum_{j=r}^p (-1)^j \binom{p}{j} \binom{j}{r} = \begin{cases} 0, & \text{if } r \neq p; \\ (-1)^p, & \text{if } r = p, \end{cases}$$

we obtain

$$\psi^{(p)}(t, n) = (-1)^p \mathcal{A}^{(p)}(t),$$

as desired. This completes the proof. \square

Corollary 9. For all integers $p \geq 0$, the following identity holds:

$$\sum_{k=1}^n k^p t^k = \frac{t}{t-1} [\mathcal{C}^{(p)}(t, n) t^n - \mathcal{C}^{(p)}(t, 0)], \quad (20)$$

where $\mathcal{C}^{(p)}(t, n) = \sum_{r=0}^p (-1)^r \mathcal{A}^{(r)}(t) \binom{p}{r} n^{p-r}$.

Proof. It is immediate from Theorem 8. \square

Definition 10. For all integers $p \geq 0$, we define the following shifted Euler polynomials $\bar{\mathcal{E}}^{(p)}(x)$ in x of degree p as

$$\bar{\mathcal{E}}^{(p)}(x) = \sum_{r=0}^p (-1)^r \mathcal{E}^{(r)} \binom{p}{r} x^{p-r}.$$

Note that $\bar{\mathcal{E}}^{(p)}(0) = (-1)^p \mathcal{E}^{(p)}$.

The first few shifted Euler polynomials are given below:

$$\bar{\mathcal{E}}^{(0)}(x) = 1$$

$$\bar{\mathcal{E}}^{(1)}(x) = x + \frac{1}{2}$$

$$\bar{\mathcal{E}}^{(2)}(x) = x^2 + x$$

$$\bar{\mathcal{E}}^{(3)}(x) = x^3 + \frac{3}{2}x^2 - \frac{1}{4}$$

$$\bar{\mathcal{E}}^{(4)}(x) = x^4 + 2x^3 - x$$

$$\bar{\mathcal{E}}^{(5)}(x) = x^5 + \frac{5}{2}x^4 - \frac{5}{2}x^2 + \frac{1}{2}$$

$$\bar{\mathcal{E}}^{(6)}(x) = x^6 + 3x^5 - 5x^3 + 3x$$

$$\bar{\mathcal{E}}^{(7)}(x) = x^7 + \frac{7}{2}x^6 - \frac{35}{4}x^4 + \frac{21}{2}x - \frac{17}{8}$$

The graph of these polynomials is given in Fig. 1.

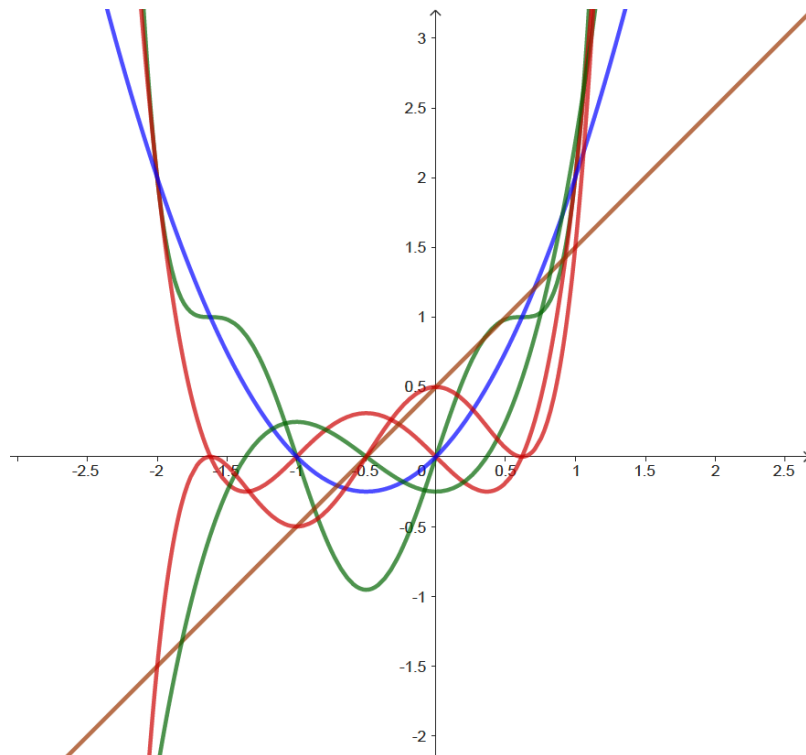


Fig. 1. Shifted Euler polynomials $\bar{\mathcal{E}}^{(p)}(x)$ for $p = 1, 2, 3, 4, 5,$ and 6 .

Corollary 11. For all integers $p \geq 0$, the following identity holds:

$$\sum_{k=1}^n (-1)^k k^p = \frac{1}{2} \left[(-1)^n \bar{\mathcal{E}}^{(p)}(n) - \bar{\mathcal{E}}^{(p)}(0) \right]. \quad (21)$$

Proof. It follows from Theorem 8 by setting $t = -1$. \square

Polynomial forms of the alternating power sum $\sum_{k=1}^n (-1)^k k^p$ for $0 \leq p \leq 10$ are given in Tab. 1.

p	$\sum_{k=1}^n (-1)^k k^p$
0	$\frac{1}{2} [(-1)^n - 1]$
1	$\frac{1}{2} \left[(-1)^n \left(n + \frac{1}{2} \right) - \frac{1}{2} \right]$
2	$\frac{1}{2} (-1)^n (n^2 + n)$
3	$\frac{1}{2} \left[(-1)^n \left(n^3 + \frac{3}{2} n^2 - \frac{1}{4} \right) + \frac{1}{4} \right]$
4	$\frac{1}{2} (-1)^n (n^4 + 2n^3 - n)$
5	$\frac{1}{2} \left[(-1)^n \left(n^5 + \frac{5}{2} n^4 - \frac{5}{2} n^2 + \frac{1}{2} \right) - \frac{1}{2} \right]$
6	$\frac{1}{2} (-1)^n (n^6 + 3n^5 - 5n^3 + 3n)$
7	$\frac{1}{2} \left[(-1)^n \left(n^7 + \frac{7}{2} n^6 - \frac{35}{4} n^4 + \frac{21}{2} n^2 - \frac{17}{8} \right) + \frac{17}{8} \right]$
8	$\frac{1}{2} (-1)^n (n^8 + 4n^7 - 14n^5 + 28n^3 - 17n)$
9	$\frac{1}{2} \left[(-1)^n \left(n^9 + \frac{9}{2} n^8 - 21n^6 + 63n^4 - \frac{153}{2} n^2 + \frac{31}{2} \right) - \frac{31}{2} \right]$
10	$\frac{1}{2} (-1)^n (n^{10} + 5n^9 - 60n^7 + 63n^5 - 255n^3 + 155n)$

Tab. 1. Polynomial forms of alternating power sums for $1 \leq p \leq 10$

Corollary 12. For all integers $p \geq 0$, the following identity holds:

$$\sum_{k=1}^n k^p 2^k = 2 \left[2^n \left(\sum_{r=0}^p (-1)^r \mathcal{F}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{F}^{(p)} \right], \quad (22)$$

where $F^{(p)}$ is the p^{th} Fubini number.

Proof. It follows from Theorem 8 by setting $t = 2$. \square

Theorem 13. For all integers $p \geq 0$, the following identity holds:

$$\begin{aligned} \sum_{k=1}^n k^p M_k &= 2 \left[2^n \left(\sum_{r=0}^p (-1)^r \mathcal{F}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{F}^{(p)} \right] \\ &\quad - \left[\frac{1}{p+1} \sum_{r=0}^p (-1)^r \binom{p+1}{r} B_r n^{p-r+1} \right], \end{aligned} \quad (23)$$

where M_k is the k^{th} Mersenne number.

Proof. Since $M_k = 2^k - 1$, we have

$$\sum_{k=1}^n k^p M_k = \sum_{k=1}^n k^p 2^k - \sum_{k=1}^n k^p.$$

Now, Eq. (23) follows from Eqs. (1) and (22). \square

Theorem 14. For all integers $p \geq 0$, the following identity holds:

$$\begin{aligned} \sum_{k=1}^n k^p J_k &= \frac{2}{3} \left[2^n \left(\sum_{r=0}^p (-1)^r \mathcal{F}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{F}^{(p)} \right] \\ &\quad - \frac{1}{6} \left[(-1)^n \left(\sum_{r=0}^p (-1)^r \mathcal{E}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{E}^{(p)} \right], \end{aligned} \quad (24)$$

where J_k is the k^{th} Jacobsthal number.

Proof. Since $J_k = \frac{1}{3}(2^k - (-1)^k)$, we have

$$\sum_{k=1}^n k^p J_k = \frac{1}{3} \left[\sum_{k=1}^p k^p 2^k - \sum_{k=1}^p (-1)^k k^p \right].$$

Now, Eq. (24) follows from Eqs. (21) and (22). \square

Setting $p = 0, 1$, and 2 in Eq. (24) yields

$$\begin{aligned} \sum_{k=1}^n J_k &= \frac{1}{6} [2^{n+2} - (-1)^n - 3], \\ \sum_{k=1}^n k J_k &= \frac{1}{12} [2n(2^{n+2} - (-1)^{n+2}) - (2^{n+3} - (-1)^{n+3}) + 9], \\ \sum_{k=1}^n k^2 J_k &= \frac{1}{6} [n^2(2^{n+2} - (-1)^{n+2}) - n(2^{n+3} - (-1)^{n+3}) + 12(2^n - 1)]. \end{aligned}$$

Rewriting these in terms of Jacobsthal and Mersenne numbers, we obtain the following lovely formulas:

$$\begin{aligned}\sum_{k=1}^n J_k &= \frac{1}{2} [J_{n+2} - 1], \\ \sum_{k=1}^n k J_k &= \frac{1}{4} [2nJ_{n+2} - J_{n+3} + 3], \\ \sum_{k=1}^n k^2 J_k &= \frac{1}{2} [n^2 J_{n+2} - nJ_{n+3} + 4M_n].\end{aligned}$$

Theorem 15. For all integers $p \geq 0$, the following identity holds:

$$\begin{aligned}\sum_{k=1}^n k^p j_k &= 2 \left[2^n \left(\sum_{r=0}^p (-1)^r \mathcal{F}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{F}^{(p)} \right] \\ &\quad + \frac{1}{2} \left[(-1)^n \left(\sum_{r=0}^p (-1)^r \mathcal{E}^{(r)} \binom{p}{r} n^{p-r} \right) - (-1)^p \mathcal{E}^{(p)} \right],\end{aligned}\tag{25}$$

where j_k is the k^{th} Jacobsthal-Lucas number.

Proof. Since $j_k = 2^k + (-1)^k$, we have

$$\sum_{k=1}^n k^p j_k = \sum_{k=1}^p k^p 2^k + \sum_{k=1}^p (-1)^k k^p.$$

Now, Eq. (25) follows from Eqs. (21) and (22). □

Setting $p = 0, 1$, and 2 , respectively, in Eq. (25) yields

$$\begin{aligned}\sum_{k=1}^n j_k &= \frac{1}{2} [2^{n+2} + (-1)^n - 5], \\ \sum_{k=1}^n k j_k &= \frac{1}{4} [2n(2^{n+2} + (-1)^{n+2}) - (2^{n+3} + (-1)^{n+3}) + 7], \\ \sum_{k=1}^n k^2 j_k &= \frac{1}{2} [n^2(2^{n+2} + (-1)^{n+2}) - n(2^{n+3} + (-1)^{n+3}) + 12(2^n - 1)].\end{aligned}$$

Rewriting these in terms of Jacobsthal-Lucas and Mersenne numbers, we obtain the following identities that bear a striking resemblance to those of Jacobsthal numbers:

$$\begin{aligned}\sum_{k=1}^n j_k &= \frac{1}{2} [j_{n+2} - 5], \\ \sum_{k=1}^n k j_k &= \frac{1}{4} [2n j_{n+2} - j_{n+3} + 7],\end{aligned}$$

$$\sum_{k=1}^n k^2 j_k = \frac{1}{2} [n^2 j_{n+2} - n j_{n+3} + 12M_n].$$

5 CONNECTIONS TO PREVIOUS WORKS

Thomas and Namboothiri [16] derived the following recurrence formula for finding the power sums:

$$\sum_{k=1}^n k^p = \frac{1}{2} \left[(n+1)^p n + \sum_{j=1}^{p-1} (-1)^j \binom{p}{j} (n+1)^{p-j} \left(\sum_{k=1}^n k^j \right) \right].$$

The main drawback of this formula is that it can be used only for odd powers p . But our formula in Theorem 3 works for all positive integer powers p .

We note that the binomial transform of the sequence $(\mathcal{E}^{(p)})_{p \geq 0}$ can be deduced from our Corollary 5. Setting $n = 1$ and $p \geq 1$ in Eq. (15) yields

$$0 = \frac{1}{2} \left[-\mathcal{E}^{(p)} - \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} \right].$$

It follows that

$$-\mathcal{E}^{(p)} = \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r}.$$

Since $\mathcal{E}^{(p)} = 0$ for all even p , we can rewrite this as

$$(-1)^p \mathcal{E}^{(p)} = \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r}.$$

Note that this is also true for $p = 0$. This is a good place to mention that Euler's identity in Eq. (4) follows from our identity in Eq. (21) as we now demonstrate.

$$\begin{aligned} \bar{\mathcal{E}}^{(p)}(x) &= \sum_{r=0}^p (-1)^r \mathcal{E}^{(r)} \binom{p}{r} x^{p-r} \\ &= \sum_{r=0}^p \left(\sum_{j=0}^r \binom{r}{j} \mathcal{E}^{(j)} \right) \binom{p}{r} x^{p-r} \end{aligned}$$

Switching the order of summation, this becomes

$$\bar{\mathcal{E}}^{(p)}(x) = \sum_{j=0}^p \left(\sum_{r=j}^p \binom{p}{r} \binom{r}{j} x^{p-r} \right) \mathcal{E}^{(j)}$$

$$\begin{aligned}
 &= \sum_{j=0}^p \binom{p}{j} (x+1)^{p-j} \mathcal{E}^{(j)} \\
 &= \mathcal{E}^{(p)}(x+1).
 \end{aligned}$$

Also, for $p \geq 1$,

$$\bar{\mathcal{E}}^{(p)}(0) = (-1)^p \mathcal{E}^{(p)} = -\mathcal{E}^{(p)} = -\mathcal{E}^{(p)}(0).$$

Thus,

$$\sum_{k=1}^n (-1)^k k^p = \frac{1}{2} [(-1)^n \mathcal{E}^{(p)}(n+1) + \mathcal{E}^{(p)}(0)],$$

which is the Euler's identity in Eq. (4), as desired.

A nice recurrence formula for $\mathcal{E}^{(p)}$ can also be deduced from the convolution identity (15) in Corollary 5. Setting $n = 1$ and $p \geq 1$ in Eq. (15) yields

$$-1 = \frac{1}{2} \left[\mathcal{E}^{(p)} - \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} 2^{p-r} \right],$$

and if we simplify this, we get

$$\mathcal{E}^{(p)} = 1 - \frac{1}{2} \sum_{r=0}^{p-1} (1 + 2^{p-r}) \binom{p}{r} \mathcal{E}^{(r)}.$$

Kim's identity in Eq. (5) can also be deduced from Corollary 5. To see this, first rewrite Eq. (14) as

$$\sum_{k=0}^{n-1} (-1)^k k^p = \frac{1}{2} \left[\mathcal{E}^{(p)} + (-1)^{n+1} \sum_{r=0}^p \mathcal{E}^{(r)} \binom{p}{r} n^{p-r} \right].$$

Now, split off the last term of the summation on the right-hand side to get

$$\sum_{k=0}^{n-1} (-1)^k k^p = \frac{1}{2} \left[\mathcal{E}^{(p)}(1 + (-1)^{n+1}) + (-1)^{n+1} \sum_{r=0}^{p-1} \mathcal{E}^{(r)} \binom{p}{r} n^{p-r} \right],$$

which is the Kim's identity in Eq. (5), as desired.

Finally, we show how to deduce Euler's general identity in Eq. (3) from Eq. (16). It is easy to prove by induction that the Eulerian polynomials $A_p(t)$ and the Eulerian rational functions $\mathcal{A}^{(p)}(t)$ are connected by the relation

$$A_p(t) = (t-1)^p \mathcal{A}^{(p)}(t),$$

for all integers $p \geq 0$. Substituting this in Eq. (16) yields

$$\begin{aligned} \sum_{k=1}^n k^p t^k &= \frac{t}{t-1} \left[\left(\sum_{r=0}^p \binom{p}{r} (-1)^r \frac{A_r(t)}{(t-1)^r} (t) n^{p-r} \right) t^n - (-1)^p \frac{A_p(t)}{(t-1)^p} \right] \\ &= -\frac{t}{1-t} \left[\left(\sum_{r=0}^p \binom{p}{r} \frac{A_r(t)}{(1-t)^r} (t) n^{p-r} \right) t^n - \frac{A_p(t)}{(1-t)^p} \right] \\ &= -t^{n+1} \sum_{r=0}^p \binom{p}{r} \frac{A_r(t)}{(1-t)^{r+1}} (t) n^{p-r} + \frac{t}{(1-t)^{p+1}} A_p(t), \end{aligned}$$

which is Euler's general identity in Eq. (3).

6 CONCLUDING REMARKS

In our work, we established the definitions of our functions $A^{(p)}(t)$ using the binomial coefficients as outlined in Eq. (10). We subsequently demonstrated that these functions are present in the convolution formulas $\sum_{k=0}^{n-1} k^p t^{n-k}$ specified in Proposition 2, which consequently resulted in their inclusion in the sum formulas $\sum_{k=1}^n t^n k^p$ presented in Theorem 3. Furthermore, we determined a recurrence formula for computing the power sums and an explicit formula for computing the alternating power sums. We have expanded the analysis to include Fubini numbers, Mersenne numbers, Jacobsthal numbers and Jacobsthal-Lucas numbers.

Several identities on infinite sums can also be deduced from our identities. For example, if $|t| < 1$ and $p \geq 1$, then by letting $n \rightarrow \infty$ in Eq. (16) and simplifying, we obtain the following interesting infinite sum:

$$\sum_{k=0}^{\infty} [(k+1)^p - k^p] t^k = (-1)^p \mathcal{A}^{(p)}(t).$$

In particular, by setting $t = \pm \frac{1}{2}$, we get

$$\frac{1^p - 0^p}{2^0} + \frac{2^p - 1^p}{2^1} + \frac{3^p - 2^p}{2^2} + \dots = (-1)^p \mathcal{A}^{(p)}\left(\frac{1}{2}\right),$$

and

$$\frac{1^p - 0^p}{2^0} - \frac{2^p - 1^p}{2^1} + \frac{3^p - 2^p}{2^2} - \dots = (-1)^p \mathcal{A}^{(p)}\left(-\frac{1}{2}\right),$$

for $p \geq 1$.

As another application, we can deduce the identity [17, Identity (19)]

$$\sum_{k=1}^n O_k = 2 - 4O_{n+2},$$

where $(O_n)_{n \geq 1}$ is the sequence of Oresme numbers defined by $O_n = \frac{n}{2^n}$ by setting $p = 1$ and $t = \frac{1}{2}$ in Eq. (16). Likewise, the identity [17, Identity (20)]

$$\sum_{k=1}^n (-1)^k O_k = \frac{4}{9} \left[-\frac{1}{2} + (-1)^{n+1} (O_{n+2} - 2O_{n+1}) \right]$$

can be deduced from Eq. (16) by setting $p = 1$ and $t = -\frac{1}{2}$. More such identities for the power sums of Oresme numbers can be deduced similarly. For example, the following identity holds:

$$\sum_{k=1}^n O_k^2 = \frac{1}{3} \left[\frac{20}{9} - O_n^2 - \frac{O_n}{3 \cdot 2^{n-3}} - \frac{5}{9 \cdot 4^{n-1}} \right].$$

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